Progress in Mathematics
290

Carlo Mantegazza

# Lecture Notes on Mean Curvature Flow

Ferran Sunyer i Balaguer
Award winning monograph

Birkhäuser

Birkhäuser

# Progress in Mathematics

Volume 290

Carlo Mantegazza

# Lecture Notes on Mean Curvature Flow

Birkhäuser

Carlo Mantegazza
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
56126 Pisa
Italy
c.mantegazza@sns.it

2010 Mathematics Subject Classification: Primary: 53C44; Secondary: 35-02, 53C21, 35K55

ISBN 978-3-0348-0144-7 e-ISBN 978-3-0348-0145-4
DOI 10.1007/978-3-0348-0145-4

Library of Congress Control Number: 2011934443



Printed on acid-free paper

Springer Basel AG is part of Springer Science+Business Media

www.birkhauser-science.com

Ferran Sunyer i Balaguer (1912–1967) was a self-taught Catalan mathematician who, in spite of a serious physical disability, was very active in research in classical mathematical analysis, an area in which he acquired international recognition. His heirs created the Fundació Ferran Sunyer i Balaguer inside the Institut d'Estudis Catalans to honor the memory of Ferran Sunyer i Balaguer and to promote mathematical research.

Each year, the Fundació Ferran Sunyer i Balaguer and the Institut d'Estudis Catalans award an international research prize for a mathematical monograph of expository nature. The prize-winning monographs are published in this series. Details about the prize and the Fundació Ferran Sunyer i Balaguer can be found at

http://ffsb.iec.cat/EN/

**This book has been awarded the Ferran Sunyer i Balaguer 2010 prize.**

INSTITVT D'ESTVDIS CATALANS MCMVII

The members of the scientific commitee of the 2010 prize were:

*Alejandro Adem*
University of British Columbia

*Hyman Bass*
University of Michigan

*Núria Fagella*
Universitat de Barcelona

*Joseph Oesterlé*
Université de Paris VI

*Joan Verdera*
Universitat Autònoma de Barcelona

Ferran Sunyer i Balaguer Prize winners since 2000:

2000 Juan-Pablo Ortega and Tudor Ratiu
*Hamiltonian Singular Reduction*, PM 222

2001 Martin Golubitsky and Ian Stewart
*The Symmetry Perspective*, PM 200

2002 André Unterberger
*Automorphic Pseudodifferential Analysis and Higher Level Weyl Calculi*, PM 209

Alexander Lubotzky and Dan Segal
*Subgroup Growth*, PM 212

2003 Fuensanta Andreu-Vaillo, Vincent Caselles and José M. Mazón
*Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*, PM 223

2004 Guy David
*Singular Sets of Minimizers for the Mumford-Shah Functional*, PM 233

2005 Antonio Ambrosetti and Andrea Malchiodi
*Perturbation Methods and Semilinear Elliptic Problems on $R^n$*, PM 240

José Seade
*On the Topology of Isolated Singularities in Analytic Spaces*, PM 241

2006 Xiaonan Ma and George Marinescu
*Holomorphic Morse Inequalities and Bergman Kernels*, PM 254

2007 Rosa Miró-Roig
*Determinantal Ideals*, PM 264

2008 Luis Barreira
*Dimension and Recurrence in Hyperbolic Dynamics*, PM 272

2009 Timothy D. Browning
*Quantitative Arithmetic of Projective Varieties*, PM 277

# Contents

# Foreword

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an $n$-dimensional smooth manifold in Euclidean space. The evolution of $M_0 = \varphi_0(M)$ by mean curvature is a smooth one-parameter family of immersions $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t), \\ \varphi(p,0) = \varphi_0(p), \end{cases}$$

where $\mathrm{H}(p,t)$ and $\nu(p,t)$ are respectively the mean curvature and the unit normal of the hypersurface $M_t = \varphi_t(M)$ at the point $p \in M$, where $\varphi_t = \varphi(\,\cdot\,, t)$.

It can be checked that $\mathrm{H}(p,t)\nu(p,t) = \Delta_{g(t)}\varphi(p,t)$, where $\Delta_{g(t)}$ is the Laplace–Beltrami operator on $M$ associated to the metric $g(t)$, induced by the immersion $\varphi_t$. Then, the mean curvature flow may be regarded as a sort of geometric heat equation, in particular it can be shown that it is a parabolic problem and has a unique solution for small time. In addition, the solutions satisfy comparison principles and derivatives estimates similar to the case of parabolic partial differential equations in Euclidean space.

On the other hand, the mean curvature flow is not really equivalent to a heat equation, since the Laplace–Beltrami operator evolves with the hypersurface itself. In particular, in contrast to the classical heat equation, this flow is described by a nonlinear (quasilinear) evolution system of partial differential equations and the solutions exist in general only in a finite time interval.

Mean curvature flow occurs in the description of the evolution of the interfaces in several multiphase physical models (see, e.g., [94, 111]); one can indeed date the "genesis" of the subject to the paper of Mullins [94]. The main reason for this is the property that it is the gradient-like flow of the *Area* functional and therefore it arises naturally in problems where a surface energy is relevant. From a physical point of view, it would be interesting also to consider the "hyperbolic" motion by mean curvature, that is, the evolution problem $\partial_t^2\varphi = \mathrm{H}\nu$, but very few results are present in the literature at the moment. Algorithms based on mean curvature flow have also been developed extensively in the field of automatic treatment of digital data, in particular of images because of the "regularizing effect" due to its parabolic nature.

Another interesting feature of this flow is its connection with certain reaction-diffusion equations, for instance

$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon} W'(u) \,,$$

where $W(u) = (u^2 - 1)^2$ (double-well potential). One can study the singular limits of the solutions of this parabolic equation when $\varepsilon$ tends to zero. Under suitable hypotheses, it can be shown that the solutions $u_\varepsilon$ with common initial data converge as $\varepsilon \to 0$ to functions which assume only the values $\pm 1$ in regions separated by boundaries evolving by mean curvature (see [6, 111]).

Further motivation for the study of mean curvature flow comes from geometric applications, in analogy with the Ricci flow of metrics on abstract Riemannian manifolds. One can use this flow as a tool to obtain classification results for hypersurfaces satisfying certain curvature conditions, to derive isoperimetric inequalities or to produce minimal surfaces. As in Hamilton's program for Ricci flow, a fundamental step in order to apply these techniques is the definition of a flow with surgeries or of a generalized (weak) notion of flow allowing one to "pass" through the singularities in a controlled way. There has been much work in this direction by means of techniques based on varifolds, level set and viscosity solutions (see [2, 7, 21, 42, 78]), till the recent results of Huisken and Sinestrari [75] that provide a surgery procedure well suited for topological conclusions.

There are striking analogies between Ricci flow and mean curvature flow. Indeed, many results hold in a similar way for both flows, and several ideas have been successfully exported from one context to the other. However, at the moment no formal way of transforming one of them into the other is known.

In these notes, we will present exclusively the "classical" parametric setting, without discussing the contributions, sometimes quite relevant, coming from other approaches, in particular, the geometric measure theory setting (see [21, 78]) and the level sets formulation (see [23, 42, 99, 120, 122, 123]).

All the manifolds, quantities and other objects we will consider are smooth, unless otherwise stated. The main tool for the analysis will be a priori estimates (pointwise and integral), very often based on a smart use of the maximum principle in the same spirit of the work of Hamilton for Ricci flow.

As of now, the study of singularities and the classification of their asymptotic shape is almost complete for some classes of evolving hypersurfaces. For others it seems difficult and quite far off. In Chapter 5 we will try to outline an up-to-date scenario of the "state of the art".

This book grew up from a collection of notes for students, and I have tried to maintain that spirit. This actually means that some discussions will be a little informal and that some points could be too detailed or even pedantic for an expert reader. With the exception of the proofs of some fundamental and deep results (listed in Appendix F), all the material is almost self-contained.

In Chapter 1 we fix the notation and we introduce the basic facts from differential geometry used throughout the book. Moreover, we define mean curvature flow, we discuss several examples and we show that any initial, smooth, compact, immersed hypersurface evolves by mean curvature at least for some positive interval of time.

In Chapter 2 we present the maximum principle and its first geometric applications to mean curvature flow, we compute the evolution equations for the relevant geometric quantities and we discuss their consequences. In particular, we establish the fact that at a singular time the curvature of the evolving hypersurface cannot stay bounded.

Chapter 3 is devoted to the analysis of *type I singularities* of mean curvature flow, that is, when the blow-up rate of the curvature at a singular time is subject to a suitable natural control. The fundamental Huisken's *monotonicity formula* will play a major role in showing that the singularities are asymptotically modeled on "homothetic shrinkers", that is, hypersurfaces that flow by mean curvature simply by homothety. The classification of such class of hypersurfaces in the special case of positive mean curvature is described with its implications.

Chapter 4 is instead concerned with *type II singularities*, that is, without the above control on the blow-up rate of the curvature. Here the goal is to show that, again when the mean curvature of the evolving hypersurface is positive, the singularities are modeled on hypersurfaces which move by translation along their mean curvature flow.

In Chapter 5 we resume many of the conclusions, moreover, we briefly discuss the recent work by Huisken and Sinestrari on *mean curvature flow with surgeries* and we list some references to open problems and research directions.

We remark that, throughout the book, special attention is given to the case of evolving curves in the plane. Very often this case requires a separate treatment and enjoys better properties than the general higher-dimensional case.

The appendices contain Polden's proof of short time existence of a solution for quasilinear parabolic PDE's on manifolds, the precise statements of some results mentioned in the book and a discussion of the Abresch–Langer classification theorem of homothetically shrinking closed curves in the plane.

## Further Literature

We definitely suggest to the reader the wonderful survey of White [121] for a general overview of the field.

An excellent introduction to mean curvature flow is provided in the monograph by Ecker [35], where many basic results and examples are collected. The second part of the book gives a fairly elementary approach to the difficult field of the regularity theory for weak solutions; in the author's opinion, it is the natural "next step" for the interested reader. Other nice general references are [34, 69, 80, 126].

Two papers which contain a survey of results on the formation of singularities for mean curvature flow (and also discuss several other geometric flows) are the ones by Huisken [68] and by Huisken and Polden [72]. It is also surely recommendable to read Sections 2 and 3 of Hamilton's fundamental paper [61]. This paper deals with Ricci flow, but many of the ideas exposed there apply to mean curvature flow as well.

Two works of central importance on weak solutions are the pioneering monograph by Brakke [21] and the memoir by Ilmanen [78]; they are more difficult reading for a beginner.

Another introductory exposition of mean curvature flow, including topics not treated in the present notes such as the connection with reaction-diffusion equations, is the one by Ambrosio [6]. The monograph by Giga [50] is also very pleasant to read and it gives a detailed account of the level sets approach to geometric evolutions.

## Acknowledgment

*I wrote these lecture notes collecting material for some introductory courses and seminars on mean curvature flow I gave in Pisa, at the SISSA in Trieste, at the Centre de Recerca Matemàtica in Barcelona and at the Institut Fourier in Grenoble. I am grateful to Gianni Dal Maso, Andrea Malchiodi, Joan Porti, Gérard Besson and Zindine Djadli for these invitations.*

*Many people helped me, in particular I wish to thank Gerhard Huisken, Carlo Sinestrari and Brian White for clarifying me several points of the original papers, Charlie Baker, Giovanni Bellettini, Francesco Bonsante, Giovanni Catino, Zindine Djadli, Alessandro Ghigi, Valeria Marcucci, Lorenzo Mazzieri, Reto Müller, Matteo Novaga and Alessandra Rosella, for a very valuable criticism and many suggestions.*

*Special thanks go to Manolo Eminenti, Annibale Magni and Luca Martinazzi for contributing to some sections of the book.*

*This book was partially written during my visits to the Institut Fourier in Grenoble, the Centre de Recerca Matemàtica in Barcelona and the Institut Henri Poincaré in Paris. In all these places I found a warm hospitality and a perfect environment to work.*

# Chapter 1

# Definition and Short Time Existence

## 1.1 Notation and Preliminaries

In this section we introduce some basic notation and facts about Riemannian manifolds and their submanifolds, a good reference is [49].

*Throughout the book, the convention of summing over the repeated indices will be adopted.*

The main objects we will consider are $n$-dimensional, complete hypersurfaces immersed in $\mathbb{R}^{n+1}$, that is, pairs $(M, \varphi)$ where $M$ is an $n$-dimensional, smooth manifold with empty boundary and $\varphi : M \to \mathbb{R}^{n+1}$ is a smooth immersion (the rank of the differential $d\varphi$ is equal to $n$ everywhere on $M$).

The manifold $M$ gets, in a natural way, a metric tensor $g$ turning it into a Riemannian manifold $(M, g)$, by pulling back the standard scalar product of $\mathbb{R}^{n+1}$ with the immersion map $\varphi$.

Taking local coordinates around $p \in M$, we have local bases of $T_pM$ and $T_p^*M$, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and 1-forms $\{dx_j\}$.

We will denote the vectors on $M$ by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, the 1-forms by $\omega = \omega_j$, that is, $\omega = \omega_j dx_j$ and a general mixed tensor by $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$, where the indices refer to the local basis.

Sometimes we will consider tensors along $M$, viewing it as a submanifold of $\mathbb{R}^{n+1}$ via the map $\varphi$, in which case we will use Greek indices to denote the components of the tensors in the canonical basis $\{e_\alpha\}$ of $\mathbb{R}^{n+1}$, for instance, given a vector field $X$ along $M$, not necessarily tangent, we will have $X = X^\alpha e_\alpha$.

The metric $g$ of $M$ extended to tensors is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l} ,$$

where $g_{ij}$ is the matrix of the coefficients of $g$ in local coordinates and $g^{ij}$ is its

C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progress in Mathematics 290,
DOI 10.1007/978-3-0348-0145-4_1, © Springer Basel AG 2011

inverse. Clearly, the norm of a tensor is

$$|T| = \sqrt{g(T,T)}\,.$$

The scalar product of $\mathbb{R}^{n+1}$ will be denoted by $\langle\cdot\,|\,\cdot\rangle$. As the metric $g$ is obtained by pulling it back via $\varphi$, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (d\varphi^*\langle\cdot\,|\,\cdot\rangle)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial\varphi}{\partial x_i}\,\middle|\,\frac{\partial\varphi}{\partial x_j}\right\rangle\,.$$

The canonical measure induced by the metric $g$ is given in a coordinate chart by $\mu = \sqrt{G}\,\mathcal{L}^n$ where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The induced covariant derivative on $(M, g)$ of a vector field $X$ and of a 1-form $\omega$ are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk}X^k\,, \qquad \nabla_j\omega_i = \frac{\partial\omega_i}{\partial x_j} - \Gamma^k_{ji}\omega_k\,,$$

where the Christoffel symbols $\Gamma^i_{jk}$ are expressed by the formula,

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right)\,.$$

The covariant derivative $\nabla T$ of a general tensor $T = T^{i_1\dots i_k}_{j_1\dots j_l}$ will be denoted by $\nabla_s T^{i_1\dots i_k}_{j_1\dots j_l} = (\nabla T)^{i_1\dots i_k}_{sj_1\dots j_l}$ (we recall that such covariant derivative is defined uniquely on the tensor algebra by imposing the Leibniz rule and the commutativity with any metric contraction).

$\nabla^m T$ will stand for the $m$th iterated covariant derivative of $T$.

The gradient $\nabla f$ of a function and the divergence $\operatorname{div} X$ of a vector field at a point $p \in M$ are defined respectively by

$$g(\nabla f(p), v) = df_p(v) \qquad \forall v \in T_pM$$

and

$$\operatorname{div} X = \operatorname{tr}\nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i}X^i + \Gamma^i_{ik}X^k\,.$$

The (rough) Laplacian $\Delta T$ of a tensor $T$ is given by

$$\Delta T = g^{ij}\nabla_i\nabla_j T\,.$$

If $X$ is a smooth vector field with compact support on $M$, as $\partial M = \emptyset$ the following *divergence theorem* holds:

$$\int_M \operatorname{div} X\,d\mu = 0\,,$$

which clearly implies

$$\int_M \Delta f \, d\mu = 0$$

for every smooth function $f : M \to \mathbb{R}$ with compact support.

Since $\varphi$ is locally an embedding in $\mathbb{R}^{n+1}$, at every point $p \in M$ we can define up to a sign a unit normal vector $\nu(p)$. Locally, we can always choose $\nu$ to be smooth.

If the hypersurface $M$ is compact and embedded, that is, the map $\varphi$ is one-to-one, the *inside* of $M$ is easily defined and we will consider $\nu$ to be the *inner pointing* unit normal vector at every point of $M$. In this case the vector field $\nu : M \to \mathbb{R}^{n+1}$ is globally smooth.

The *second fundamental form* $\mathrm{A} = h_{ij}$ of $M$ is the symmetric 2-form defined as

$$h_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle,$$

and the *mean curvature* $\mathrm{H}$ is the trace of $\mathrm{A}$, that is $\mathrm{H} = g^{ij} h_{ij}$. Despite its name, $\mathrm{H}$ is the *sum* of the eigenvalues of the second fundamental form, not their average mean (some few authors actually define $\mathrm{H}/n$ as the mean curvature).

*Remark* 1.1.1. Notice that since $\nu$ is defined up to a sign, the same is true for $\mathrm{A}$. Instead, the *vector-valued second fundamental form* $h_{ij}\nu$, which is a 2-form with values in $\mathbb{R}^{n+1}$, is uniquely defined.

With our choice of $\nu$ as the inner pointing unit normal, the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has a positive definite second fundamental form and positive mean curvature and the same holds for every strictly *convex* hypersurface of $\mathbb{R}^{n+1}$.

*We advise the reader that throughout the book, by a little abuse of terminology, we will say that a hypersurface is convex when its second fundamental form is nonnegative definite, strictly convex when it is positive definite. If the hypersurface is embedded, convexity in this sense is equivalent to the usual definition that the hypersurface bounds a convex subset of the Euclidean space.*

The linear map $W_p : T_pM \to T_pM$ given by $W_p(v) = h^i_j(p) v^j \frac{\partial}{\partial x_i}$ is called the *Weingarten operator* and its eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ the *principal curvatures* at the point $p \in M$. It is easy to see that $\mathrm{H} = \lambda_1 + \cdots + \lambda_n$ and $|\mathrm{A}|^2 = \lambda_1^2 + \cdots + \lambda_n^2$.

**Exercise 1.1.2.** Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $\varphi(x) = (x, f(x))$, then we have

$$g_{ij} = \delta_{ij} + f_i f_j \,, \qquad \nu = -\frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}}, \qquad h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1 + |\nabla f|^2}} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{(\sqrt{1 + |\nabla f|^2})^3} = \mathrm{div}\left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right)$$

where $f_i = \partial_i f$ and $\mathrm{Hess} f$ is the Hessian of the function $f$.

**Exercise 1.1.3.** Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the *zero set* of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, with $\nabla f \neq 0$ on such level set, then we have

$$\mathrm{H} = \frac{\Delta f}{|\nabla f|} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^3} = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right).$$

The following Gauss–Weingarten relations will be fundamental,

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma_{ij}^k \frac{\partial \varphi}{\partial x_k} + h_{ij}\nu\,, \qquad \frac{\partial}{\partial x_j}\nu = -h_{jl} g^{ls} \frac{\partial \varphi}{\partial x_s}\,. \tag{1.1.1}$$

Actually, they express the fact that $\nabla^M = \nabla^{\mathbb{R}^{n+1}} - \mathrm{A}\nu$. We recall that considering $M$ locally as a regular submanifold of $\mathbb{R}^{n+1}$, we have $\nabla^M_X Y = (\nabla^{\mathbb{R}^{n+1}}_X \widetilde{Y})^M$ where the sign $^M$ denotes the projection on the tangent space to $M$ and $\widetilde{Y}$ is a local extension of the field $Y$ in a local neighborhood $\Omega \subset \mathbb{R}^{n+1}$ of $\varphi(M)$.
Notice that, by these relations, it follows that

$$\Delta\varphi = g^{ij}\nabla^2_{ij}\varphi = g^{ij}\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial\varphi}{\partial x_k}\right) = g^{ij}h_{ij}\nu = \mathrm{H}\nu\,. \tag{1.1.2}$$

By straightforward computations, we can see that the Riemann tensor, the Ricci tensor and the scalar curvature can be expressed by means of the second fundamental form as follows,

$$\begin{aligned}
\mathrm{R}_{ijkl} &= g\Big(\nabla^2_{ji}\frac{\partial}{\partial x_k} - \nabla^2_{ij}\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\Big) = h_{ik}h_{jl} - h_{il}h_{jk}\,,\\
\mathrm{Ric}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}\,h_{ij} - h_{il}g^{lk}h_{kj}\,,\\
\mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = g^{ij}g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}^2 - |\mathrm{A}|^2\,.
\end{aligned}$$

Hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned}
\nabla_i\nabla_j X^s - \nabla_j\nabla_i X^s &= \mathrm{R}_{ijkl}g^{ks}X^l = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ks}X^l\,,\\
\nabla_i\nabla_j \omega_k - \nabla_j\nabla_i \omega_k &= \mathrm{R}_{ijkl}g^{ls}\omega_s = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ls}\omega_s\,.
\end{aligned}$$

The symmetry properties of the covariant derivative of A are given by the Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij} \tag{1.1.3}$$

which imply the *Simons' identity* (see [106]),

$$\Delta h_{ij} = \nabla_i\nabla_j \mathrm{H} + \mathrm{H}\,h_{il}g^{ls}h_{sj} - |\mathrm{A}|^2 h_{ij}\,. \tag{1.1.4}$$

We will write $T * S$, following Hamilton [55], to mean a tensor formed by a sum of terms each one of them obtained by contracting some indices of the pair $T$, $S$ with the metric $g_{ij}$ and/or its inverse $g^{ij}$.

A very useful property of such $*$-product is that

$$|T * S| \leq C\,|T|\,|S|$$

where the constant $C$ depends only on the algebraic "structure" of $T * S$.

Sometimes we will need the $n$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, and we will denote it by $\mathcal{H}^n$.

*We advise the reader that, in all computations, the constants could vary between different formulas and from one line to another.*

## 1.2 First Variation of the Area Functional

Given an immersion $\varphi : M \to \mathbb{R}^{n+1}$ of a smooth hypersurface in $\mathbb{R}^{n+1}$, we consider the Area functional

$$\mathrm{Area}(\varphi) = \int_M d\mu$$

where $\mu$ is the canonical measure associated to the metric $g$ induced by the immersion.

In this section we are going to analyze the first variation of such functional which is nothing else than the volume of the hypersurface.

We consider a smooth one-parameter family of immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, outside of a compact set $K \subset M$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$.
Defining the field $X = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ along $M$ (as a submanifold of $\mathbb{R}^{n+1}$) we see that $X$ is zero outside $K$; we call such a field the *infinitesimal generator* of the variation $\varphi_t$.
We compute

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \varphi_t}{\partial x_i} \,\Big|\, \frac{\partial \varphi_t}{\partial x_j}\right\rangle\Big|_{t=0} \\
&= \left\langle \frac{\partial X}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j} \,\Big|\, \frac{\partial \varphi}{\partial x_i}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_i}\right\rangle - 2\left\langle X \,\Big|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \\
&= \frac{\partial}{\partial x_i}\left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_i}\right\rangle - 2\Gamma_{ij}^k\left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_k}\right\rangle - 2h_{ij}\langle X \,|\, \nu\rangle\,,
\end{aligned}$$

where $X^M$ is the tangent component of the field $X$ and we used the Gauss–Weingarten relations (1.1.1) in the last step.
Letting $\omega$ be the 1-form defined by $\omega(Y) = g(d\varphi^*(X^M), Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma_{ij}^k \omega_k - 2h_{ij}\langle X \,|\, \nu\rangle = \nabla_i \omega_j + \nabla_j \omega_i - 2h_{ij}\langle X \,|\, \nu\rangle\,.$$

Hence, using the formula $\partial_t \det A(t) = \det A(t)\mathrm{Trace}[A^{-1}(t)\partial_t A(t)]$, we get

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\,\bigg|_{t=0} &= \frac{\sqrt{\det(g_{ij})}g^{ij}\frac{\partial}{\partial t}g_{ij}\big|_{t=0}}{2}\\
&= \frac{\sqrt{\det(g_{ij})}g^{ij}(\nabla_i\omega_j + \nabla_j\omega_i - 2h_{ij}\langle X\,|\,\nu\rangle)}{2}\\
&= \sqrt{\det(g_{ij})}(\operatorname{div} X^M - \mathrm{H}\langle X\,|\,\nu\rangle)\,.
\end{aligned}$$

If the Area of the immersion $\varphi$ is finite, the same holds for all the $\varphi_t$, as they are compact deformations of $\varphi$. Assuming that the compact $K$ is contained in a single coordinate chart, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\,\bigg|_{t=0} &= \frac{\partial}{\partial t}\int_K d\mu_t\,\bigg|_{t=0}\\
&= \frac{\partial}{\partial t}\int_K \sqrt{\det(g_{ij})}\,d\mathcal{L}^n\,\bigg|_{t=0}\\
&= \int_K \frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\,\bigg|_{t=0}\,d\mathcal{L}^n\\
&= \int_K (\operatorname{div} X^M - \mathrm{H}\langle X\,|\,\nu\rangle)\sqrt{\det(g_{ij})}\,d\mathcal{L}^n\\
&= \int_M (\operatorname{div} X^M - \mathrm{H}\langle X\,|\,\nu\rangle)\,d\mu\\
&= -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu
\end{aligned}$$

where we used the fact that $X$ is zero outside $K$ and in the last step we applied the divergence theorem. Notice that all the integrals are well defined because we are actually integrating only on the compact set $K$.
In the case that $K$ is contained in several charts, the same conclusion follows from a standard argument using a partition of unity.

**Proposition 1.2.1.** *The first variation of the* Area *functional depends only on the normal component of the infinitesimal generator $X = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ of the variation $\varphi_t$, precisely*

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\bigg|_{t=0} = -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\,.$$

*Clearly, such dependence is linear.*

Given any immersion $\varphi : M \to \mathbb{R}^{n+1}$ and any vector field $X$ with compact support along $M$, we can always construct a variation with infinitesimal generator $X$ as $\varphi_t(p) = \varphi(p) + tX(p)$. It is easy to see that for $|t|$ small the map $\varphi_t$ is still a smooth immersion.

Hence, as the hypersurfaces which are critical points of the Area functional must satisfy $\int_M \mathrm{H}\langle X \mid \nu\rangle\, d\mu = 0$ for every field $X$ with compact support, they must have $\mathrm{H} = 0$ everywhere, that is, zero mean curvature (and conversely). This is the well-known definition of the so-called *minimal surfaces.*

As the quantity $-\mathrm{H}\nu$ can be interpreted as the *gradient* of the Area functional (be careful here, the measure $\mu$ is varying with the immersion, we are not computing the gradient with respect to some fixed $L^2$-structure on the space of immersions of $M$ in $\mathbb{R}^{n+1}$), we can consider the motion of a hypersurface by minus this gradient, that is, the *mean curvature flow.* So, one looks for hypersurfaces moving with velocity $\mathrm{H}\nu$ at every point. This means choosing, among all the velocity functions with fixed $L^2(\mu)$-norm equal to $\left(\int_M \mathrm{H}^2\, d\mu\right)^{1/2}$, the one such that the Area of hypersurface decreases most rapidly.

This idea is quite natural and arises often in studying the dynamics of models of physical situations where the energy is given by the "Area" of the interfaces between the phases of a system. Moreover, as the Area functional is the simplest (in terms of derivatives of the parametrization) geometric functional, that is, invariant by isometries of $\mathbb{R}^{n+1}$ and diffeomorphisms of $M$, the motion by mean curvature is the simplest *variational* geometric flow for immersed hypersurfaces. Other geometric functionals (for instance, depending on the next simpler geometric invariant, the curvature) generally produce a first variation of order higher than two in the derivatives of the parametrization and a relative higher order PDE's system.

One can consider second-order flows where the velocity of the motion is related to different functions of the curvature, like the Gauss flow of surfaces, for instance, where the velocity is given by $\mathrm{G}\nu$ ($\mathrm{G}$ is the Gauss curvature of $M$, that is, $\mathrm{G} = \det \mathrm{A}$) or more complicated flows, but these evolutions are usually not variational, they do not arise as "gradients" (in the above sense) of geometric functionals (see Section 1.6).

## 1.3 The Mean Curvature Flow

**Definition 1.3.1.** Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an $n$-dimensional manifold. The mean curvature flow of $\varphi_0$ is a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ such that setting $\varphi(p, t) = \varphi_t(p)$ the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth solution of the following system of PDE's

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p, t) = \mathrm{H}(p, t)\nu(p, t), \\ \varphi(p, 0) = \varphi_0(p), \end{cases} \tag{1.3.1}$$

where $\mathrm{H}(p, t)$ and $\nu(p, t)$ are respectively the mean curvature and the unit normal of the hypersurface $\varphi_t$ at the point $p \in M$.

*Remark* 1.3.2. Notice that even if the unit normal vector is defined up to a sign, the field $\mathrm{H}(p, t)\nu(p, t)$ is independent of such choice.

Using equation (1.1.2), this system can be rewritten in the appealing form

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi$$

but, despite its formal analogy with the heat equation, actually, it is a second-order, *quasilinear* and *degenerate*, parabolic system, as the Laplacian is the one associated to the evolving hypersurfaces at time $t$,

$$\Delta \varphi(p,t) = \Delta_{g(p,t)} \varphi(p,t) = g^{ij}(p,t) \nabla_i^{g(p,t)} \nabla_j^{g(p,t)} \varphi(p,t)$$

and both $g$ and $\nabla^g$ depend on the first derivatives of $\varphi$. Moreover, this operator is degenerate, as its symbol (the symbol of the linearized operator) admits zero eigenvalues due to the invariance of the Laplacian by diffeomorphisms, see [48] for details.

Like the Area functional, the flow is obviously invariant by rotations and translations, or more generally under any isometry of $\mathbb{R}^{n+1}$. Moreover, if $\varphi(p,t)$ is a mean curvature flow and $\Psi : M \to M$ is a diffeomorphism, then $\widetilde{\varphi}(p,t) = \varphi(\Psi(p),t)$ is still a mean curvature flow. This last property can be reread as "the flow is invariant under reparametrization", suggesting that the important objects in the flow are actually the subsets $M_t = \varphi(M,t)$ of $\mathbb{R}^{n+1}$.

The problem also satisfies the following parabolic invariance under rescaling (consequence of the property $\mathrm{Area}(\lambda\varphi) = \lambda^n \mathrm{Area}(\varphi)$, for any $n$-dimensional immersion); if $\varphi(p,t)$ is a mean curvature flow of $\varphi_0$ and $\lambda > 0$, then $\widetilde{\varphi}(p,t) = \lambda\varphi(p, \lambda^{-2}t)$ is a mean curvature flow of the initial hypersurface $\lambda\varphi_0$.

During the flow the Area of the hypersurfaces (which is the natural energy of the problem) is nonincreasing, indeed by the same computation for the first variation, we have

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t) = \frac{\partial}{\partial t}\int_M d\mu_t = -\int_M \mathrm{H}^2\, d\mu_t\,.$$

This clearly implies the estimate

$$\int_0^{T_{\max}} \int_M \mathrm{H}^2\, d\mu_t \leq \mathrm{Area}(\varphi_0)$$

in the maximal time interval $[0, T_{\max})$ of smooth existence of the flow.

**Exercise 1.3.3.** By means of this last inequality, try to get an estimate from above for the maximal time of smooth existence $T_{\max}$ for closed curves in $\mathbb{R}^2$ and compact *surfaces* in $\mathbb{R}^3$.

**Proposition 1.3.4 (Geometric Invariance under Tangential Perturbations).** *If a smooth family of immersions $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ satisfies the system of PDE's*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) + X(p,t), \\ \varphi(p,0) = \varphi_0(p), \end{cases}$$

*where $X$ is a time-dependent smooth vector field along $M$ such that $X(p,t)$ belongs to $d\varphi_t(T_pM)$ for every $p \in M$ and every time $t \in [0,T)$, then, locally around any point in space and time, there exists a family of reparametrizations (smoothly time-dependent) of the maps $\varphi_t$ which satisfies system* (1.3.1).
*If the hypersurface $M$ is compact, one can actually find uniquely a family of* global *reparametrizations of the maps $\varphi_t$ as above for every $t \geq 0$, leaving the initial immersion $\varphi_0$ unmodified and satisfying system* (1.3.1).
*Conversely, if a smooth family of moving hypersurfaces $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ can be globally reparametrized for $t \geq 0$ in order that it moves by mean curvature, then the map $\varphi$ has to satisfy the system above for some time-dependent vector field $X$ with $X(p,t) \in d\varphi_t(T_pM)$.*

*Proof.* First we assume that $M$ is compact; we will produce a smooth global parametrization of the evolving sets in order to check the definition.
By the tangency hypothesis, the time-dependent vector field on $M$ given by $Y(q,t) = -[d\varphi_t]^{-1}(X(q,t))$ is globally well defined and smooth.
Let $\Psi : M \times [0,T) \to M$ be a smooth family of diffeomorphisms of $M$ with $\Psi(p,0) = p$ for every $p \in M$ and

$$\frac{\partial}{\partial t}\Psi(p,t) = Y(\Psi(p,t),t)\,. \tag{1.3.2}$$

This family exists, is unique and smooth, by the existence and uniqueness theorem for ODE's on the compact manifold $M$.
Considering the reparametrizations $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$, one has

$$\begin{aligned}
\frac{\partial\widetilde{\varphi}}{\partial t}(p,t) &= \frac{\partial\varphi}{\partial t}(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))\Big(\frac{\partial\Psi}{\partial t}(p,t)\Big)\\
&= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t) + X(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))(Y(\Psi(p,t),t))\\
&= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t) + X(\Psi(p,t),t)\\
&\qquad - d\varphi_t(\Psi(p,t))([d\varphi_t(\Psi(p,t))]^{-1}(X(\Psi(p,t),t)))\\
&= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t)\\
&= \widetilde{\mathrm{H}}(p,t)\widetilde{\nu}(p,t)\,.
\end{aligned}$$

Hence, $\widetilde{\varphi}$ satisfies system (1.3.1) and $\widetilde{\varphi}_0 = \varphi_0$.
Conversely, this computation also shows that if $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$ satisfies system (1.3.1), the family of diffeomorphisms $\Psi : M \times [0,T) \to M$ must solve equation (1.3.2), hence, it is unique if we assume $\Psi(\,\cdot\,,0) = \mathrm{Id}_M$ in order that the map $\varphi_0$ is unmodified.
In the noncompact case, we have to work locally in space and time, solving the above system of ODE's in some positive interval of time in an open subset $\Omega \subset M$ with compact closure, then obtaining a solution of system (1.3.1) in a possibly smaller open subset of $\Omega$ and some interval of time.

Assume now that the reparametrized map $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$ is a mean curvature flow. Differentiating, we get

$$\begin{aligned}\frac{\partial\widetilde{\varphi}}{\partial t}(p,t) &= \frac{\partial\varphi}{\partial t}(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))\Big(\frac{\partial\Psi}{\partial t}(p,t)\Big)\\ &= \widetilde{\mathrm{H}}(p,t)\widetilde{\nu}(p,t)\\ &= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t)\end{aligned}$$

that is,

$$\frac{\partial\varphi}{\partial t}(q,t) = \mathrm{H}(q,t)\nu(q,t) - d\varphi_t(q)\Big(\frac{\partial\Psi}{\partial t}(\Psi_t^{-1}(q),t)\Big)\,,$$

for every $q \in M$ and $t \in [0,T)$. Then, the last statement of the proposition follows by setting $X(q,t) = -d\varphi_t(q)\Big(\frac{\partial\Psi}{\partial t}(\Psi_t^{-1}(q),t)\Big)$. $\square$

**Corollary 1.3.5.** *If a smooth family of hypersurfaces $\varphi_t = \varphi(\,\cdot\,,t)$ satisfies $\langle\partial_t\varphi|\nu\rangle = \mathrm{H}$, then it can be everywhere locally reparametrized to a mean curvature flow. If $M$ is compact, this can be done uniquely by global reparametrizations, without modifying $\varphi_0$.*

*Remark* 1.3.6. A short way to state the previous proposition and corollary is to say that the tangential component of the velocity of the points of the hypersurface does not affect the global "shape" during the motion.
This is particularly meaningful in the case that system (1.3.1) has a unique solution, for instance when $M$ is compact, as we will see in Theorem 1.5.1 in the next section.

By this invariance property one is led to speak of mean curvature flow of hypersurfaces considering them as subsets of $\mathbb{R}^{n+1}$ and forgetting their parametrizations. This is clear in the case of embedded hypersurfaces, where the identification of $(M, g(t))$ with the images of the embeddings $\varphi_t(M)$ is immediate, but it also works for nonembedded hypersurfaces as every immersion is locally an embedding.
We give then a more geometric, alternative definition of the mean curvature flow. In the sequel of the book it will be clear by the context which one we are using.

**Definition 1.3.7.** We still say that a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, for $t \in [0,T)$, is a mean curvature flow if locally at every point, in space and time, there exists a family of reparametrizations which satisfies system (1.3.1).

Proposition 1.3.4 expresses the substantial equivalence between this definition (Eulerian point of view) and Definition 1.3.1 (Lagrangian point of view).

**Exercise 1.3.8 (Motion of Graphs).** Show that if the smooth hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$, moving by mean curvature, are locally graphs on some open subset $\Omega$ of the hyperplane $\langle e_1,\dots,e_n\rangle \subset \mathbb{R}^{n+1}$, that is, if we have a smooth function $f : \Omega \times [0,T) \to \mathbb{R}$, such that $\varphi(p,t) = (x_1(p),\dots,x_n(p), f(x_1(p),\dots,x_n(p),t))$, then

$$\partial_t f = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} = \sqrt{1 + |\nabla f|^2}\,\mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right).$$

On the other hand, if we have a function $f$ satisfying the above parabolic equation, then its graph is a hypersurface moving by mean curvature (according to Definition 1.3.7).

**Exercise 1.3.9 (Motion of Level Sets).** Assume that for every time $t \in [0, T)$ the image $\varphi_t(M)$ of the smooth, embedded hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$, moving by mean curvature, is the *zero set* of $f_t = f(\,\cdot\,, t)$, where $f : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ is a smooth function and zero is a regular value of $f_t$ for every $t \in [0, T)$. Then at all the points $x \in \mathbb{R}^{n+1}$ and times $t \in [0, T)$ such that $f(x, t) = 0$ there holds

$$\partial_t f = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^2} = |\nabla f|\,\mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right).$$

Conversely, if we have a smooth function $f$ satisfying the above parabolic equation, *every regular* level set of $f(\,\cdot\,, t)$ is a hypersurface moving by mean curvature (according to Definition 1.3.7).

**Exercise 1.3.10 (Distance Functions).** Compute the evolution equation satisfied by the signed distance function $d_{M_t} : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ at the points $x \in M_t = \varphi_t(M)$, if the compact and embedded, smooth hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$ move by mean curvature.
The signed distance function is the function which coincides with the distance in the region "outside" a hypersurface and with minus the distance in the "inside" region (show that it is smooth in a tubular neighborhood of the hypersurface).

**Exercise 1.3.11 (Brakke's Definition [21]).** Show that a smooth family of compact and embedded hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$ satisfies

$$\frac{d}{dt}\int_{\varphi_t(M)} f \, d\mathcal{H}^n \leq \int_{\varphi_t(M)} \big(\mathrm{H}\langle \nabla f \,|\, \nu\rangle - \mathrm{H}^2 f\big)\, d\mathcal{H}^n,$$

for every *positive* function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, if and only if the hypersurfaces are moving by mean curvature flow (according to Definition 1.3.7).
In the formula $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

## 1.4 Examples

Spheres and cylinders are the easiest and actually some of the few nontrivial explicitly computable examples of mean curvature flows (minimal surfaces are trivial examples as they are not moving at all, satisfying $\mathrm{H} = 0$).

Let us consider a sphere of radius $R$ which, by the translation invariance of the flow, we can assume to be centered at the origin of $\mathbb{R}^{n+1}$. A right guess is that at every time the hypersurface remains a sphere and the mean curvature flow simply changes its radius $R(t)$, this is actually true by the uniqueness theorem in the next section. As the mean curvature is everywhere equal to $n/R$ and since we chose the inner pointing unit normal, the evolution equation for the radius of the sphere is simply $R'(t) = -n/R(t)$ with $R(0) = R$. Indeed, if we set $M = \mathbb{S}^n$ and $\varphi(p,t) = R(t)\varphi_0(p)$, being $\varphi_0$ the standard immersion of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we have

$$R'(t)\varphi_0(p) = \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) = -n\varphi_0(p)/R(t)\,,$$

which is an ODE that can be easily integrated to get $R(t) = \sqrt{R^2 - 2nt}$.
At time $T_{\max} = R^2/(2n)$ the sphere shrinks to a point so the flow becomes singular; this is the maximal time of existence. We can then write the evolution of the radius also as $R(t) = \sqrt{2n(T_{\max} - t)}$.
During the flow the norm of the second fundamental form evolves as

$$|\mathrm{A}(t)| = \sqrt{n}/R(t) = \frac{1}{\sqrt{2(T_{\max} - t)}}\,.$$

Other examples are given by the cylinders $\mathbb{S}^m(R)\times\mathbb{R}^{n-m}$. In general, we can see that if $\varphi : M \times [0,T) \to \mathbb{R}^{m+1}$ is a mean curvature flow of an $m$-dimensional hypersurface $M$ of $\mathbb{R}^{m+1}$, then the map $\widetilde{\varphi} : (M\times\mathbb{R}^{n-m})\times[0,T) \to \mathbb{R}^{m+1}\times\mathbb{R}^{n-m} = \mathbb{R}^{n+1}$, defined by $\widetilde{\varphi}(p,s,t) = (\varphi(p,t), s)$, is a mean curvature flow of the immersion of the product manifold $M \times \mathbb{R}^{n-m}$ in $\mathbb{R}^{n+1}$.
Then, by the above discussion, these cylinders evolve homothetically as $\mathbb{S}^m(R(t))\times\mathbb{R}^{n-m}$, with $R(t) = \sqrt{R^2 - 2mt}$ and collapse to the subspace $\{0\} \times \mathbb{R}^{n-m}$ at time $T_{\max} = R^2/(2m)$. Again, the norm of the second fundamental form satisfies $|\mathrm{A}(t)| = \frac{1}{\sqrt{2(T_{\max}-t)}}$.

Spheres and cylinders are special examples of *homothetically shrinking* flows, that is, hypersurfaces that simply move by contraction during the evolution by mean curvature.

**Proposition 1.4.1.** *If an initial hypersurface* $\varphi_0 : M \to \mathbb{R}^{n+1}$ *satisfies* $\mathrm{H}(p) + \lambda\langle\varphi_0(p) - x_0\,|\,\nu_0(p)\rangle = 0$ *at every point* $p \in M$ *for some constant* $\lambda > 0$ *and* $x_0 \in \mathbb{R}^{n+1}$*, then it generates a homothetically shrinking mean curvature flow* (*according to Definition* 1.3.7)*.*
*Conversely, if* $\varphi : M\times[0,T) \to \mathbb{R}^{n+1}$ *is a homothetically shrinking mean curvature flow* (*according to Definition* 1.3.7) *around some point* $x_0 \in \mathbb{R}^{n+1}$ *in a maximal time interval, then either* $\mathrm{H}$ *is identically zero or*

$$\mathrm{H}(p,t) + \frac{\langle\varphi(p,t) - x_0\,|\,\nu(p,t)\rangle}{2(T-t)} = 0\,,$$

*for every point* $p \in M$ *and time* $t \in [0,T)$*.*

*Proof.* If the condition is satisfied, we consider the homothetically shrinking flow

$$\varphi(p,t) = x_0 + \sqrt{1-2\lambda t}(\varphi_0(p) - x_0)$$

and we see that

$$\langle \partial_t \varphi(p,t) \,|\, \nu(p,t)\rangle = -\frac{\lambda\langle \varphi_0(p) - x_0 \,|\, \nu(p,t)\rangle}{\sqrt{1-2\lambda t}} = \frac{\mathrm{H}(p,0)}{\sqrt{1-2\lambda t}} = \mathrm{H}(p,t)\,,$$

as $\nu(p,t) = \nu_0(p)$. Hence, by Corollary 1.3.5, this is a mean curvature flow of the initial hypersurface $\varphi_0$, according to Definition 1.3.7.

Conversely, if the homothetically shrinking evolution

$$\varphi(p,t) = x_0 + f(t)(\varphi_0(p) - x_0)$$

is a mean curvature flow, for some positive smooth function $f : [0,T) \to \mathbb{R}$ with $f(0) = 1$, $\lim_{t\to T} f(t) = 0$ and $f'(t) \leq 0$, by Corollary 1.3.5 we have $\langle \partial_t\varphi \,|\, \nu\rangle = \mathrm{H}$, hence

$$\begin{aligned}
\mathrm{H}(p,0) &= f(t)\mathrm{H}(p,t)\\
&= f(t)\langle \partial_t\varphi(p,t) \,|\, \nu(p,t)\rangle\\
&= f(t)f'(t)\langle \varphi_0(p) - x_0 \,|\, \nu(p,t)\rangle\\
&= f'(t)\langle \varphi(p,t) - x_0 \,|\, \nu(p,t)\rangle\,.
\end{aligned}$$

If $\mathrm{H} \neq 0$ at some point, as $\nu(p,t) = \nu_0(p)$ we have that $f(t)f'(t)$ is equal to some constant $C$ for every $t \in [0,T)$, combining the first and the third line of the above formula. Hence, $f(t) = \sqrt{2Ct+1}$ as $f(0) = 1$ and since $\lim_{t\to T} f(t) = 0$, we conclude $f(t) = \sqrt{1-t/T}$. The thesis then follows from the first and last line of the formula. □

We underline that again $|\mathrm{A}(t)| \sim \frac{1}{\sqrt{2(T_{\max}-t)}}$.

Up to a rigid motion and rescaling, solving the above structural equation is equivalent to solving $\mathrm{H} + \langle \varphi_0 \,|\, \nu_0\rangle = 0$.

In the special case of curves in $\mathbb{R}^2$, supposing $\gamma$ parametrized by its arclength $s$, the unit tangent vector is given by $\tau = \gamma_s$ and the unit normal by $\nu = \mathrm{R}\gamma_s$, where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of $\pi/2$. Then, such equation becomes

$$\gamma_{ss} = k = -\langle \gamma \,|\, \nu\rangle = -\langle \gamma \,|\, \mathrm{R}\gamma_s\rangle\,,$$

hence one can find the homothetically shrinking curves by integrating this ODE. The only embedded solutions are $\mathbb{S}^1$ and the lines through the origin of $\mathbb{R}^2$ (see Appendix E), but there are also several other nonembedded closed curves found by Abresch and Langer [1] that classified all the possible solutions, see also the work of Epstein and Weinstein [40].

Finding homothetically shrinking hypersurfaces when the dimension is higher than one is difficult (see the discussion in [80]). It is known that besides the "standard" examples given by the hyperplanes through the origin, the spheres and the cylinders, there exists a homothetically shrinking, embedded torus in $\mathbb{R}^3$, found by Angenent [17] (it seems that Grayson was the first to suggest its existence, see [67]). Moreover, there is numerical evidence that higher genus surfaces in $\mathbb{R}^3$ could also exist, see Chopp [24] and Angenent, Chopp and Ilmanen [18] (see also [80]). Finally, a tentative strategy to produce new examples is being carried on in some recent papers by Nguyen [95, 98].

One can also look for homothetically *expanding* hypersurfaces, which are characterized by the same equation $\mathrm{H}+\lambda\langle\varphi_0\,|\,\nu_0\rangle = 0$ but with a negative constant $\lambda$. These cannot be compact, as one can see easily by looking at the point of maximum distance from the origin of $\mathbb{R}^{n+1}$.
As an example, every angle less than $\pi$ in $\mathbb{R}^2$ contains a convex, unbounded, homothetically expanding curve under the curvature flow, asymptotic to the edges of the angle (see also the discussion in [21, Appendix C]). A classification of graph solutions of the above equation with at most linear growth can be found in [114].

Another notable family of hypersurfaces moving by mean curvature are the ones generating *translating* flows, these are hypersurfaces that during the motion do not change their shape but simply move in a fixed direction with constant velocity.

**Proposition 1.4.2.** *If an initial hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$ satisfies $\mathrm{H}(p) = \langle v\,|\,\nu_0(p)\rangle$ at every point $p \in M$ for some constant vector $v \in \mathbb{R}^{n+1}$, then it generates a translating mean curvature flow* (*according to Definition* 1.3.7) *with constant velocity $v$.*
*Conversely, if $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is a translating mean curvature flow* (*according to Definition* 1.3.7) *then there exists a vector $v \in \mathbb{R}^{n+1}$* (*which is the velocity of the motion*) *such that $\mathrm{H}(p,t) = \langle v\,|\,\nu(p,t)\rangle$ for every point $p \in M$ and $t \in [0,T)$.*

*Proof.* If the condition is satisfied, we consider the translating flow $\varphi(p,t) = \varphi_0(p) + tv$ and we see that

$$\langle \partial_t\varphi(p,t)\,|\,\nu(p,t)\rangle = \langle v\,|\,\nu(p,t)\rangle = \langle v\,|\,\nu_0(p)\rangle = \mathrm{H}(p,0) = \mathrm{H}(p,t)\,,$$

as $\nu(p,t) = \nu_0(p)$. Hence, by Corollary 1.3.5, this is a mean curvature flow of the initial hypersurface $\varphi_0$, according to Definition 1.3.7.

Conversely, if the translating flow $\varphi(p,t) = \varphi_0(p) + w(t)$ is a mean curvature flow, for some smooth, time-dependent vector $w : [0,T) \to \mathbb{R}^{n+1}$ with $w(0) = 0$, by Corollary 1.3.5 we have $\langle\partial_t\varphi\,|\,\nu\rangle = \mathrm{H}$, hence

$$\langle \partial_t\varphi(p,t)\,|\,\nu(p,t)\rangle = \langle w'(t)\,|\,\nu(p,t)\rangle = \mathrm{H}(p,t) = \mathrm{H}(p,0)\,.$$

Suppose that varying $p$ in $M$, the image of the unit normal is a subset of $\mathbb{R}^{n+1}$ whose span is the whole $\mathbb{R}^{n+1}$, then as $\nu(p,t) = \nu_0(p)$ if we differentiate in time the above equality we get $\langle w''(t) \,|\, \nu_0(p) \rangle = 0$ which implies that $w''(t) = 0$ for every $t$ and $w'(t)$ constant. Then, letting $w' = v$, we have the thesis.

In the case that the span of the image of the unit normal is not the whole $\mathbb{R}^{n+1}$, all the tangent spaces $T_pM$ to $\varphi_0$ have a common nontrivial vector subspace $L \subset \mathbb{R}^{n+1}$. Decomposing $w(t) = l(t) + z(t)$ with $l(t) \in L$ and $z(t) \in L^\perp$ we have $l(0) = z(0) = 0$ and $z''(t) = 0$ by the above argument, as $l'(t)$, $l''(t) \in L$ and $z'(t)$, $z''(t) \in L^\perp$. Hence, $z'(t)$ is constant and

$$\mathrm{H}(p,t) = \langle w'(t) \,|\, \nu(p,t) \rangle = \langle z'(t) \,|\, \nu(p,t) \rangle = \langle v \,|\, \nu(p,t) \rangle$$

where we set $v = z'$.
By Proposition 1.3.4 (see also Remark 1.3.6) the translating flow $\widetilde{\varphi}(p,t) = \varphi_0(p) + tv$ coincides (as sets) with the flow $\varphi$ as $\partial_t \widetilde{\varphi} = v$, $\partial_t \varphi = w'(t) = l'(t) + v$ and $l'(t) \in L \subset T_pM$ for every $p \in M$ and $t \in [0,T)$. $\square$

In the special case of curves in $\mathbb{R}^2$, if $\gamma$ is parametrized by its arclength $s$ and the unit normal is $\nu = \mathrm{R}\gamma_s$ as before, the above equation becomes

$$\gamma_{ss} = k = \langle v \,|\, \nu \rangle = \langle v \,|\, \mathrm{R}\gamma_s \rangle \,,$$

where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of $\pi/2$. By integrating such ODE, one can see that the only possible translating curve is given (up to homotheties and rigid motions) by the graph of the function $x = -\log\cos y$ in the interval $(-\pi/2, \pi/2)$, which was called the *grim reaper* by Grayson [51].

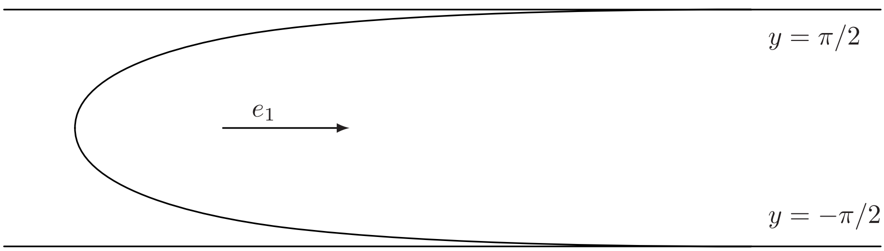


Figure 1: The *grim reaper* moving with constant velocity $e_1$.

In higher dimension, for every fixed vector $v \in \mathbb{R}^{n+1}$ there is a unique rotationally symmetric, strictly convex hypersurface (which is actually an entire graph) moving by translation under the mean curvature flow. Indeed, looking for a convex graph over a domain in $\mathbb{R}^n$ (identified with the hyperplane $\{x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$), translating in the $e_{n+1}$ direction with unit speed, one has to find a convex function $f : \Omega \to \mathbb{R}$ such that

$$\Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} = 1$$

and $f(0) = \nabla f(0) = 0$, where $\mathrm{Hess} f$ is the Hessian of $f$.

Imposing rotational symmetry around the origin $f(x) = f(\rho)$ with $\rho = |x|$, this problem becomes the ODE

$$f_{\rho\rho} + \frac{(n-1)f_\rho}{\rho} - \frac{f_{\rho\rho}f_\rho^2}{1+f_\rho^2} = 1\,,$$

that is,

$$f_{\rho\rho} = (1+f_\rho^2)\left(1 - \frac{(n-1)f_\rho}{\rho}\right) \tag{1.4.1}$$

with $\lim_{\rho\to 0} f(\rho) = \lim_{\rho\to 0} f'(\rho) = 0$ for a convex function $f : \mathbb{R}^+ \to \mathbb{R}$.
When $n = 1$ the solution of this ODE gives the *grim reaper*; when $n > 1$ there is only one solution, defined on all $\mathbb{R}^+$ and growing quadratically at infinity. This solution provides the only rotationally symmetric, convex, translating hypersurface moving by mean curvature, up to homotheties and rigid motions.

**Exercise 1.4.3.** Show the claimed properties of the solution of such ODE.

In the paper by Wang [119] it is proved that in dimension two every convex and translating flow must be rotationally symmetric and that in every dimension larger than two there exist examples which are not rotationally symmetric and also non-entire solutions of the above PDE (actually, convex solutions defined in strips of $\mathbb{R}^n$). See also [121, page 536] and [54].
Recently, Nguyen [96, 97] exhibited some new nonconvex, embedded examples of translating hypersurfaces, with a trident-like shape at “large scales”.

Finally, there are also rotating (or rotating and dilating/contracting) flows, see [76, 81] and [26, Section 2.2], like the unbounded spiral in the plane called the *Yin–Yang* curve that rotates during its motion by curvature, depicted in the paper [3] by Altschuler.

We have seen that the homothetically shrinking hypersurfaces cannot “live” forever; at some maximal time $T_{\max} > 0$ the map $\varphi$ becomes singular. This is a common fact to any compact initial hypersurface, as we will see in the next chapter in Corollary 2.2.5.
There are two possible reasons why this happens: the first is analytic, the function $\varphi$ could stop being smooth, usually because some derivatives of $\varphi$ are not bounded as $t \to T_{\max}$; the second reason is geometric, when the map $\varphi$ stops being an immersion, that is, $d\varphi_t$ becomes singular (not one-to-one) at some point of $M$ and time $t = T_{\max}$. We will see that in both cases the curvature of the evolving hypersurface has to become unbounded, that is, if the second fundamental form A stays uniformly bounded till $T_{\max}$ then this latter cannot be the maximal time of existence of a smooth flow (Proposition 2.4.6).

In all the above self-similar examples, either the flow is smoothly defined for every positive time or at some finite time the hypersurface instantly completely vanishes. This is quite a special behavior, indeed, in general the singularities develop only in some regions of the evolving hypersurface. An example of a more generic and "concrete" singularity is a nonembedded cardioid curve in the plane with a small loop: at some time the small loop has shrunk while the rest of the curve has remained smooth and a cusp has developed (see [14, 15, 16]).

In this example the initial curve was not embedded (we will see in the sequel why this was necessary); another example, this time embedded, of what can happen at the singular time is the *dumbbell* surface [52], or *standard neckpinch*.

Consider a long thin cylinder (the neck) in $\mathbb{R}^3$ smoothly connecting symmetrically two large spheres at its ends (it is possible to construct an example of such a surface also with $\mathrm{H} > 0$), then, during the flow one can guess that the cylinder, which has a large positive mean curvature, shrinks faster than the two big spheres at its ends which share instead a small curvature, having a large radius. Then, the cylinder collapses at some time and the hypersurface tends to the symmetric union of two "water drops" joined at the vertices of their cusps.

The existence of surfaces with this behavior under the mean curvature flow was first proved rigorously by Grayson [52]. Another similar example was worked out by Huisken in [67]. The following simple argument can be found in [17]: consider Angenent's homothetically shrinking torus, mentioned above, surrounding the "cylindrical" part of a symmetric dumbbell surface and a couple of spheres "inside" the two side balls, in a way that neither the torus nor the two spheres touch the dumbbell. We will see in Theorem 2.2.1 that then the four surfaces cannot touch each other during all their flows, as they have no intersections at the initial time. If the dumbbell is chosen in such a way that the torus (which is shrinking homothetically to a point) has an extinction time $T$ smaller than the one of the two "inside" spheres, the dumbbell must develop some kind of singularity before $T$ since it is "squeezed" by the torus, but it cannot vanish as it has still to "contain" the two spheres.

Our last example is the standard torus in $\mathbb{R}^3$ obtained by rotating around the $z$ axis a very small circle with a center far enough from such axis (if the circle is small enough with respect to its distance from the $z$ axis, this torus has $\mathrm{H} > 0$). During the evolution by mean curvature, the circle tends to shrink before the whole torus has time to collapse, hence, by the rotational symmetry which is maintained during the flow, one expects that at the maximal time the torus develops a circle of singularities around the $z$ axis. This suggests that in general the limit shape of an evolving hypersurface at a singular time can be quite wild (some results on the "size" of the singular sets have been obtained by White [120]).

These last examples motivated the large research in literature about suitable weak solutions of the mean curvature flow, in order to define a generalized evolution even after a singular time. In a physical model, indeed, the interface could continue the evolution even after such time, possibly in a nonsmooth way. In the dumbbell situation above, for instance, we intuitively expect that the surface splits in two parts, each of them moving independently after the splitting. Also for topological applications, in the same spirit of Hamilton's program for the Ricci flow, it is important to be able to continue the flow after any singular time

until the hypersurface (or all the parts in which it separates) converges to some known limit.
Weak solutions of the mean curvature flow have been introduced by many authors in different ways; among the others, we recall the definition by Brakke [21], based on geometric measure theory and the ones by Chen, Giga, Goto [23] and Evans, Spruck [42] based on the level sets formulation and the theory of viscosity solutions.

## 1.5 Short Time Existence of the Flow

**Theorem 1.5.1.** *For any initial, smooth and compact hypersurface of* $\mathbb{R}^{n+1}$ *given by an immersion* $\varphi_0 : M \to \mathbb{R}^{n+1}$*, there exists a unique, smooth solution of system* (1.3.1) *in some positive time interval.*
*Moreover, the solution continuously depends on the initial immersion* $\varphi_0$ *in* $C^\infty$*.*

*Remark* 1.5.2. In literature this result was proved in several ways:

- Gage and Hamilton used the Nash–Moser inverse function theorem, actually a very strong tool for the existence of solutions of parabolic systems of PDE's (we recall that we are dealing with a *degenerate quasilinear* system), see [48]. To the author's knowledge, this is the first published proof of smooth existence for short time, moreover it works in general for the motion by mean curvature of a compact submanifold of any codimension immersed in a Riemannian manifold.
- Evans and Spruck in [42, 43] showed the existence of a smooth solution of the PDE satisfied by the distance function from an embedded hypersurface moving by mean curvature (see also [88]).
- Huisken and Polden in [72] (or in the Ph.D. Thesis of Polden [102]) reduced the degenerate parabolic system to a *nondegenerate* parabolic *equation*, representing the evolving hypersurfaces as graphs over the initial one in a tubular neighborhood of this latter (this line is mentioned also in [17] and [38]).

  This is the proof that we are going to show below and which is the most natural one in the "classical" approach to mean curvature flow. Moreover it is quite elastic to be adapted to a very general class of flows of hypersurfaces and to be generalized to any codimension and any Riemannian ambient space (see [39]).

Another possibility is to use some version of the so-called *DeTurck's trick* (see [33]), coupling the mean curvature flow with another flow, in order to eliminate the diffeomorphism invariance of the problem (see [126] and [22]).

Moreover, there are also many existence proofs of generalized evolutions by mean curvature (after introducing weak definitions of hypersurfaces) of nonregular or possibly "wild" subsets of $\mathbb{R}^{n+1}$; some even allow these latter to be merely closed sets in the Euclidean space. It should be said that, considering nonsmooth subsets in these generalized definitions of the flow, the uniqueness fails in several situations.

We mention some of these approaches:

- The use of theory of viscosity solutions to study the PDE satisfied by the function $f$ in the formulation of the motion via level sets mentioned in Exercise 1.3.9, exploited by Evans and Spruck in [42, 43] and Chen, Giga and Goto in [23].
- The study by Soner [109] of the evolution equation satisfied by the signed distance function from an embedded hypersurface moving by mean curvature (see Exercise 1.3.10) by means of barrier comparison arguments and Perron's method.
- The varifold approach of Brakke using geometric measure theory, see [21] (a hint of Brakke's weak definition of mean curvature flow is given in Exercise 1.3.11).
- Almgren, Taylor and Wang discretization-minimization procedure in [2].
- Ilmanen's approximation in [77, 78].

*Remark* 1.5.3. One can show that the mean curvature flow shares a kind of the usual regularizing property of parabolic equations, for instance, any $C^2$ initial hypersurface becomes analytic at every positive time, in the sense that it is not the map $\varphi_t$ which becomes analytic, but the image hypersurface $\varphi_t(M) \subset \mathbb{R}^{n+1}$, that is, it admits an analytic reparametrization.

Moreover, with the right definition, one can let evolve a hypersurface with corners or other singularities and these latter immediately vanish, see for instance [14, 16] and [38, 118].

*Proof.* We follow Huisken and Polden in [72].

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact $n$-dimensional manifold. For the moment we assume that this hypersurface is embedded, hence the inner pointing unit normal vector field $\nu_0$ is globally defined and smooth.

We look for a smooth solution $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ of the parabolic problem

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p, t) = \mathrm{H}(p, t)\nu(p, t), \\ \varphi(p, 0) = \varphi_0(p) \end{cases}$$

for some $T > 0$.

Since we are interested in a solution for short time, we can forget about the immersion condition ($d\varphi_t$ nonsingular) as it will follow automatically by the smoothness of the solution and by the fact that $\varphi_0$ is a compact immersion, when $t$ is close to zero.

Keeping in mind Proposition 1.3.4 and Corollary 1.3.5, if we find a smooth solution of the problem

$$\begin{cases} \left\langle \frac{\partial}{\partial t}\varphi(p,t) \,\middle|\, \nu(p,t)\right\rangle = \mathrm{H}(p,t), \\ \varphi(p,0) = \varphi_0(p), \end{cases} \tag{1.5.1}$$

then we are done.

We consider the regular tubular neighborhood $\Omega = \{x \in \mathbb{R}^{n+1} \,|\, d(x, \varphi_0(M)) < \varepsilon\}$, which exists for $\varepsilon > 0$ small enough. By regular we mean that the map $\Psi : M \times (-\varepsilon, \varepsilon) \to \Omega$ defined as $\Psi(p,s) = \varphi_0(p) + s\nu_0(p)$ is a diffeomorphism.

Any small deformation of $\varphi_0(M)$ inside $\Omega$ can be represented as the graph of a "height" function $f$ over $\varphi_0(M)$ and conversely, to any small function $f : M \to \mathbb{R}$ we can associate the hypersurface $M_f \subset \Omega$ given by $\varphi(p) = \varphi_0(p) + f(p)\nu_0(p)$. We want to compute now the equation for a smooth function $f$, time-dependent, in order that $\varphi$ satisfies system (1.5.1).

Obviously, as $f(\,\cdot\,, 0)$ gives the hypersurface $\varphi_0$, we have $f(p,0) = 0$ for every $p \in M$.

First we compute the metric and the normal of the perturbed hypersurfaces, we set $f_i = \partial_i f$ then,

$$\begin{aligned} g_{ij}(p,t) &= \left\langle \frac{\partial\varphi(p,t)}{\partial x_i} \,\middle|\, \frac{\partial\varphi(p,t)}{\partial x_j} \right\rangle \\ &= \left\langle \frac{\partial\varphi_0}{\partial x_i} + f_i\nu_0 - f h_i^k(0)\frac{\partial\varphi_0}{\partial x_k} \,\middle|\, \frac{\partial\varphi_0}{\partial x_j} + f_j\nu_0 - f h_j^l(0)\frac{\partial\varphi_0}{\partial x_l} \right\rangle \\ &= \left\langle \frac{\partial\varphi_0}{\partial x_i} - f h_i^k(0)\frac{\partial\varphi_0}{\partial x_k} \,\middle|\, \frac{\partial\varphi_0}{\partial x_j} - f h_j^l(0)\frac{\partial\varphi_0}{\partial x_l} \right\rangle + f_i f_j \\ &= g_{ij}(p,0) - 2f(p,t)h_{ij}(p,0) + f^2(p,t)h_{ik}g^{kl}h_{lj}(p,0) + f_i(p,t)f_j(p,t) \end{aligned}$$

where we used Gauss–Weingarten equations (1.1.1).

The vectors

$$\frac{\partial\varphi(p,t)}{\partial x_i} = \frac{\partial\varphi_0}{\partial x_i} + f_i(p,t)\nu_0(p) - f(p,t)h_i^k(p,0)\frac{\partial\varphi_0}{\partial x_k}$$

generate the tangent space, hence the normal $\nu(p,t)$ is given by

$$\begin{aligned} \nu(p,t) &= \frac{\nu_0(p) - \left\langle \nu_0(p) \,\middle|\, \frac{\partial\varphi(p,t)}{\partial x_i} \right\rangle g^{ij}(p,t)\frac{\partial\varphi(p,t)}{\partial x_j}}{\left|\nu_0(p) - \left\langle \nu_0(p) \,\middle|\, \frac{\partial\varphi(p,t)}{\partial x_i} \right\rangle g^{ij}(p,t)\frac{\partial\varphi(p,t)}{\partial x_j}\right|} \\ &= \frac{\nu_0(p) - f_i(p,t)g^{ij}(p,t)\frac{\partial\varphi(p,t)}{\partial x_j}}{\left|\nu_0(p) - f_i(p,t)g^{ij}(p,t)\frac{\partial\varphi(p,t)}{\partial x_j}\right|}\,. \end{aligned}$$

Notice that the normal, the metric and thus its inverse depend only on first space derivatives of the function $f$. Moreover, as $f(p,0) = \nabla f(p,0) = 0$, everything is

smooth and since $M$ is compact, when $t$ is small the denominator of the above expression for the normal is uniformly bounded below away from zero (actually it is close to one).
Then, we find the second fundamental form,

$$\begin{aligned}h_{ij}(p,t) = &\Big\langle \nu(p,t) \,\Big|\, f_{ij}(p,t)\nu_0 + \frac{\partial^2\varphi_0}{\partial x_i \partial x_j} - f_i(p,t)h^k_j(0)\frac{\partial\varphi_0}{\partial x_k} \\ &\qquad - f_j(p,t)h^l_i(0)\frac{\partial\varphi_0}{\partial x_l} + f(p,t)\frac{\partial^2\nu_0}{\partial x_i\partial x_j}\Big\rangle \\ = &\langle \nu(p,t)\,|\, f_{ij}(p,t)\nu_0\rangle + P_{ij}(p, f(p,t), \nabla f(p,t))\end{aligned}$$

where $P_{ij}$ is a smooth form when $f$ and $\nabla f$ are small, hence for $t$ small.
Computing in normal coordinates around $p \in M$ with respect to the metric $g(t)$, the mean curvature is then given by

$$\begin{aligned}\mathrm{H}(p,t) &= g^{ij}(p,t)h_{ij}(p,t) \\ &= \langle \nu(p,t)\,|\,\nu_0(p)\rangle f_{ij}(p,t)g^{ij}(p,t) + P_{ij}(p, f(p,t), \nabla f(p,t))g^{ij}(p,t) \\ &= \langle \nu(p,t)\,|\,\nu_0(p)\rangle \Delta_{g(t)} f + P(p, f(p,t), \nabla f(p,t))\,,\end{aligned}$$

where $P$ is a smooth function, assuming that $f$ and $\nabla f$ are small.
We are finally ready to write down the condition $\langle \partial_t\varphi\,|\,\nu\rangle = \mathrm{H}$ in terms of the function $f$,

$$\begin{aligned}\frac{\partial f(p,t)}{\partial t}\langle \nu_0(p)\,|\,\nu(p,t)\rangle &= \langle \partial_t\varphi(p,t)\,|\,\nu(p,t)\rangle \\ &= \mathrm{H}(p,t) \\ &= \langle \nu(p,t)\,|\,\nu_0(p)\rangle \Delta_{g(t)} f + P(p, f(p,t), \nabla f(p,t))\,,\end{aligned}$$

thus, if we divide both sides by $\langle \nu(p,t)\,|\,\nu_0(p)\rangle$, which we can assume to be nonzero for a small positive time, we get

$$\begin{aligned}\frac{\partial f(p,t)}{\partial t} &= \Delta_{g(t)} f + \frac{P(p, f(p,t), \nabla f(p,t))}{\langle \nu(p,t)\,|\,\nu_0(p)\rangle} \\ &= \Delta_{g(t)} f + Q(p, f(p,t), \nabla f(p,t))\end{aligned}$$

where $Q(p,\,\cdot\,,\,\cdot\,)$ is a smooth function when its arguments are small. Moreover, as the coefficients of $\Delta_{g(t)}$ smoothly converge to the coefficients of $\Delta_{g(0)}$ as $t \to 0$, for $t$ small, the operators $\Delta_{g(t)}$ are uniformly strictly elliptic.
Then, if the smooth function $f : M \times [0,T) \to \mathbb{R}$ solves the following partial differential *equation* (before we had to deal with a *system* of PDE's)

$$\begin{cases}\frac{\partial f}{\partial t}(p,t) = \Delta_{g(t)} f + Q(p, f, \nabla f) \\ f(p,0) = 0\end{cases} \tag{1.5.2}$$

then $\varphi(p,t) = \varphi_0(p) + f(p,t)\nu_0(p)$ is a solution of system (1.5.1) for the initial compact embedding $\varphi_0$, in a positive time interval.

Conversely, if we have a mean curvature flow $\varphi$ of $\varphi_0$, for small time the hypersurfaces $\varphi_t$ are embedded in the tubular neighborhood $\Omega$ of $\varphi_0(M)$, then the function $f(p,t) = \pi_{(-\varepsilon,\varepsilon)}[\Psi^{-1}(\varphi(p,t))]$ is smooth and $f(p,0) = \pi_{(-\varepsilon,\varepsilon)}[\Psi^{-1}(\varphi_0(p))] = \pi_{(-\varepsilon,\varepsilon)}[(p,0)] = 0$, where $\pi_{(-\varepsilon,\varepsilon)}$ is the projection map on the second factor of $M \times (-\varepsilon,\varepsilon)$, hence, by the above computations $f$ must solve problem (1.5.2).

This PDE is a quasilinear strictly parabolic equation, by what we said about the uniform ellipticity of $\Delta_{g(t)}$. In particular it is not degenerate (in some sense, passing to the height function $f$ we killed the degeneracy of systems (1.3.1) and (1.5.1)) hence, we can apply the (almost standard) theory of quasilinear parabolic PDE's. The proof of a general theorem about existence, uniqueness and continuous dependence of a solution for a class of problems including (1.5.2) can be found again in [72] (see Appendix A).

Using the unique solution $f$ of problem (1.5.2) we consider the associated map $\varphi = \varphi_0 + f\nu_0$, we possibly restrict the time interval in order that $\varphi_t$ are all immersions and we apply Corollary 1.3.5 to reparametrize globally the hypersurfaces in a unique way in order to get a solution of system (1.3.1). This association is one-to-one, as long as one remains inside the regular tubular neighborhood $\Omega$, hence, existence, uniqueness, smoothness and dependence on the initial datum of a solution of system (1.3.1) follows from the analogous result for quasilinear parabolic PDE's.

If the hypersurface is not embedded, that is, it has self-intersections, since locally every immersion is an embedding, we only need a little bit more care in the definition of the height function associated to a mean curvature flow (a regular global tubular neighborhood is missing), in order to see that the correspondence between a map $\varphi$ and its height function $f$ is still a bijection, then the same argument gives the conclusion also in the nonembedded case. $\square$

*Remark* 1.5.4. This theorem gives the existence and uniqueness of the mean curvature flow in the case of a *compact* initial hypersurface. The noncompact case is more involved, as one needs estimates on the initial hypersurface (like similarly, on the initial datum in order to deal with the heat equation in all $\mathbb{R}^n$) to have existence in some positive interval of time. One possibility is to assume a uniform control on the norm of the second fundamental form of the initial hypersurface (see [38]).

Actually, by means of interior estimates (see Appendix B) Ecker and Huisken in [38] showed that a *uniform local Lipschitz condition* on a hypersurface is sufficient to guarantee short time existence. Another remarkable consequence of their work is the fact that the entire graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ has a smooth global mean curvature evolution for every time, remaining always a graph, see [37, 38] (notice that the same statement is not true for the heat equation).

Similar interior estimates, depending only on a local bound on the value of a *function* (not on its gradient) whose graph is moving by mean curvature, were obtained by Colding and Minicozzi [28].

The uniqueness of the evolution by mean curvature of a noncompact initial hypersurface is another delicate point, like for the heat equation in $\mathbb{R}^n$. One possibility is to restrict the class of “admissible” evolutions, in order to have uniqueness, to the ones with a uniform bound (local in time) on the second fundamental form along the flow; see anyway [22] for the strongest result in this context.

*Remark* 1.5.5. The apparent loss of uniqueness of the flow if one considers the evolution by mean curvature of a hypersurface given as a subset of $\mathbb{R}^{n+1}$ (see Remark 1.3.6), due to the arbitrariness in choosing the parametrization, can actually be dealt with by noticing that the system (1.3.1) is invariant by reparametrization. Hence, even if possibly the immersions describing the hypersurfaces at time $t$ are different, the hypersurfaces in $\mathbb{R}^{n+1}$ are however the same, that is, the flow is “geometrically” unique.

## 1.6 Other Second-order Flows

Let $\mathrm{S} = \mathrm{S}(\lambda_1, \dots, \lambda_n)$ be a symmetric function of the principal curvatures. Given an initial smooth immersion $\varphi_0 : M \to \mathbb{R}^{n+1}$ of the $n$-dimensional manifold $M$, one can consider the more general evolution problem

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{S}(p,t)\nu(p,t), \\ \varphi(p,0) = \varphi_0(p), \end{cases} \tag{1.6.1}$$

where $\mathrm{S}(p,t)$ is a short way to denote the value of S associated to the curvatures $\lambda_1, \dots, \lambda_n$ of the hypersurface $\varphi_t$ at the point $p \in M$. Besides the mean curvature flow, which is given by the choice $\mathrm{S} = \lambda_1 + \cdots + \lambda_n$, other studied cases are the *Gauss curvature flow*, where $\mathrm{S} = \lambda_1\lambda_2 \cdots \lambda_n = \det \mathrm{A}$ is the Gauss curvature and the *inverse mean curvature flow* considering $\mathrm{S} = (\lambda_1 + \cdots + \lambda_n)^{-1} = 1/\mathrm{H}$ (see [71], for instance).

For all these flows, we have the following existence result [72], which follows along the same line of Theorem 1.5.1.

**Theorem 1.6.1.** *Let $M$ be compact and assume that at every point $p \in M$ we have*

$$\frac{\partial \mathrm{S}}{\partial \lambda_i}(\lambda_1(p), \dots, \lambda_n(p)) > 0, \qquad i = 1, \dots, n, \tag{1.6.2}$$

*for an initial hypersurface $\varphi_0$, then system* (1.6.1) *has a unique smooth solution in some positive time interval.*

It can be checked that condition (1.6.2) is equivalent to the parabolicity of the PDE’s system at the initial time. In the case of the mean curvature flow such condition is satisfied for every initial hypersurface. For other flows one possibly needs to restrict the initial hypersurfaces to certain classes. For instance, the above result ensures the well-posedness of the Gauss curvature flow only when all the eigenvalues $\lambda_i$ are positive everywhere on the initial hypersurface, that is, when it is strictly convex.

# Chapter 2

# Evolution of Geometric Quantities

In studying the long term behavior of solutions of parabolic equations and systems, in particular in the analysis of singularities, a basic step is always to obtain a priori estimates. These can be integral or pointwise; the main tool in order to get these latter is the *maximum principle*, in particular in the context of mean curvature flow.

## 2.1 Maximum Principle

**Theorem 2.1.1.** *Assume that $g(t)$, for $t \in [0,T)$, is a family of Riemannian metrics on a manifold $M$, with a possible boundary $\partial M$, such that the dependence on $t$ is smooth.*
*Let $u : M \times [0,T) \to \mathbb{R}$ be a smooth function satisfying*

$$\partial_t u \leq \Delta_{g(t)} u + \langle X(p,u,\nabla u,t) \,|\, \nabla u \rangle_{g(t)} + b(u)$$

*where $X$ and $b$ are respectively a continuous vector field and a locally Lipschitz function in their arguments.*
*Then, suppose that for every $t \in [0,T)$ there exists a value $\delta > 0$ and a compact subset $K \subset M \setminus \partial M$ such that at every time $t' \in (t-\delta, t+\delta) \cap [0,T)$ the maximum of $u(\,\cdot\,,t')$ is attained at least at one point of $K$ (this is clearly true if $M$ is compact without boundary).*
*Setting $u_{\max}(t) = \max_{p\in M} u(p,t)$ we have that the function $u_{\max}$ is locally Lipschitz, hence differentiable at almost every time $t \in [0,T)$ and at every differentiability time,*

$$\frac{du_{\max}(t)}{dt} \leq b(u_{\max}(t))\,.$$

*As a consequence, if $h : [0,T') \to \mathbb{R}$ is a solution of the ODE*

$$\begin{cases} h'(t) = & b(h(t)), \\ h(0) = & u_{\max}(0), \end{cases}$$

*for $T' \leq T$, then $u \leq h$ in $M \times [0,T')$.*

C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progress in Mathematics 290,
DOI 10.1007/978-3-0348-0145-4_2, 

*Moreover, if $M$ is connected and at some time $\tau \in (0, T')$ we have $u_{\max}(\tau) = h(\tau)$, then $u = h$ in $M \times [0, \tau]$, that is, $u(\,\cdot\,, t)$ is constant in space.*

**Corollary 2.1.2.** *Under the same hypotheses, when $M$ is connected and the function $b$ is nonpositive (in particular if it is identically zero), if the maximum of $u$ is nondecreasing in a time interval $I$, the function $u$ is constant in $M \times I$.*

The first part of the theorem is a consequence of the following lemma. The last claim, the *strong* maximum principle, is more involved, see the book of Landis [82] for a proof and the extensive discussion in [27, Chapter 12].

**Lemma 2.1.3 (Hamilton's Trick [56]).** *Let $u : M \times (0, T) \to \mathbb{R}$ be a $C^1$ function such that for every time $t$, there exists a value $\delta > 0$ and a compact subset $K \subset M \setminus \partial M$ such that at every time $t' \in (t - \delta, t + \delta)$ the maximum $u_{\max}(t') = \max_{p \in M} u(p, t')$ is attained at least at one point of $K$.*

*Then, $u_{\max}$ is a locally Lipschitz function in $(0, T)$ and at every differentiability time $t \in (0, T)$ we have*

$$\frac{du_{\max}(t)}{dt} = \frac{\partial u(p, t)}{\partial t}$$

*where $p \in M \setminus \partial M$ is any interior point where $u(\,\cdot\,, t)$ gets its maximum.*

*Proof.* Fixing $t \in (0, T)$, we have $\delta > 0$ and $K$ as in the hypotheses, hence on $K \times (t - \delta, t + \delta)$ the function $u$ is Lipschitz with some Lipschitz constant $C$. Consider a value $0 < \varepsilon < \delta$, then we have

$$u_{\max}(t + \varepsilon) = u(q, t + \varepsilon) \leq u(q, t) + \varepsilon C \leq u_{\max}(t) + \varepsilon C\,,$$

for some $q \in K$, hence,

$$\frac{u_{\max}(t + \varepsilon) - u_{\max}(t)}{\varepsilon} \leq C\,.$$

Analogously,

$$u_{\max}(t) = u(p, t) \leq u(p, t + \varepsilon) + \varepsilon C \leq u_{\max}(t + \varepsilon) + \varepsilon C\,,$$

for some $p \in K$, hence,

$$\frac{u_{\max}(t) - u_{\max}(t + \varepsilon)}{\varepsilon} \leq C\,.$$

With the same argument, considering $-\delta < \varepsilon < 0$, we conclude that $u_{\max}$ is a locally Lipschitz function in $(0, T)$, hence differentiable at almost every time.

Suppose that $t$ is one of such times; let $p$ be a point in the nonempty set $\{p \in M \setminus \partial M \,|\, u(p, t) = u_{\max}(t)\}$.

By Lagrange's theorem, for every $0 < \varepsilon < \delta$, $u(p, t+\varepsilon) = u(p,t) + \varepsilon \frac{\partial u(p,\xi)}{\partial t}$ for some $\xi$, hence

$$u_{\max}(t+\varepsilon) \geq u(p, t+\varepsilon) = u_{\max}(t) + \varepsilon \frac{\partial u(p,\xi)}{\partial t}\,,$$

which implies, as $\varepsilon > 0$,

$$\frac{u_{\max}(t+\varepsilon) - u_{\max}(t)}{\varepsilon} \geq \frac{\partial u(p,\xi)}{\partial t}\,.$$

Sending $\varepsilon$ to zero, we get $u'_{\max}(t) \geq \frac{\partial u(p,t)}{\partial t}$.
If instead we choose $-\delta < \varepsilon < 0$ we get

$$\frac{u_{\max}(t+\varepsilon) - u_{\max}(t)}{\varepsilon} \leq \frac{\partial u(p,\xi)}{\partial t}$$

and when $\varepsilon \to 0$, we have $u'_{\max}(t) \leq \frac{\partial u(p,t)}{\partial t}$. Thus, we are done. □

**Exercise 2.1.4.** Prove that the conclusion of the lemma holds also if the function $u$ is merely locally Lipschitz, provided that all the derivatives involved in the computations there exist.

*Proof of Theorem* 2.1.1 – *First Part.* By the previous lemma, the function $u_{\max}$ is locally Lipschitz and letting $t$ be a differentiability time of $u_{\max}$, we have, choosing any $p \in M \setminus \partial M$ such that $u(p,t) = u_{\max}(t)$,

$$\begin{aligned} u'_{\max}(t) = \frac{\partial u(p,t)}{\partial t} &\leq \Delta_{g(t)} u + \langle X(p, u, \nabla u, t) \,|\, \nabla u \rangle_{g(t)} + b(u(p,t)) \\ &\leq b(u(p,t)) \\ &= b(u_{\max}(t))\,. \end{aligned}$$

Let now $h : [0, T') \to \mathbb{R}$ be as in the hypothesis. We define, for $\varepsilon > 0$, the approximating functions $h_\varepsilon : [0, T'') \to \mathbb{R}$ to be the maximal solutions of the family of ODE's

$$\begin{cases} h'_\varepsilon(t) = & b(h_\varepsilon(t)), \\ h_\varepsilon(0) = & u_{\max}(0) + \varepsilon\,. \end{cases}$$

It is easy to see that, as the function $b$ is locally Lipschitz, then $\lim_{\varepsilon\to 0} h_\varepsilon = h$ uniformly on $[0, T' - \delta]$ for any $\delta > 0$. Suppose that at some positive time $u_{\max} > h_\varepsilon$ and set $\overline{t} > 0$ to be the positive infimum of such times (at time zero $u_{\max}(0) = h_\varepsilon(0) - \varepsilon$). Then, $u_{\max}(\overline{t}) = h_\varepsilon(\overline{t})$ and, setting $H_\varepsilon = h_\varepsilon - u_{\max}$, at every differentiability point of $u_{\max}$ in the interval $[0, \overline{t})$ we have $H_\varepsilon(0) = \varepsilon > 0$ and

$$H'_\varepsilon(t) \geq b(h_\varepsilon(t)) - b(u_{\max}(t)) \geq -C(h_\varepsilon(t) - u_{\max}(t)) = -CH_\varepsilon(t)$$

where $C > 0$ is a local Lipschitz constant for $b$.

Then, $(\log H_\varepsilon)'(t) \geq -C$ and integrating, $\log H_\varepsilon|_0^t \geq -Ct$, that is, $H_\varepsilon(t) \geq H_\varepsilon(0)e^{-Ct} = \varepsilon e^{-Ct}$. In particular, if $t \to \overline{t}$, we conclude $H_\varepsilon(\overline{t}) \geq \varepsilon e^{-C\overline{t}} > 0$ which is in contradiction with $H_\varepsilon(\overline{t}) = 0$. Hence, $u_{\max}(t) \leq h_\varepsilon(t)$ for every $t \in [0, T' - \delta)$ and sending $\varepsilon$ to zero, $u_{\max}(t) \leq h(t)$ for every $t \in [0, T' - \delta)$. As $\delta > 0$ was arbitrary, we conclude the proof of the first part of the theorem. □

**Exercise 2.1.5.** When the function $u_{\max}$ is not differentiable at $t$, one can still actually say something using the upper derivative, that is the lim sup of the incremental ratios; we call this operator $d^+$. Prove that

$$\frac{d^+ u_{\max}(t)}{dt} = \sup_{\{p \in M \,|\, u(p,t) = u_{\max}(t)\}} \frac{\partial u(p,t)}{\partial t}\,.$$

Roughly speaking, the sup and the upper derivative operators can be interchanged.

The same holds for the inf and the lower derivative defined analogously.

What can be said about the left/right derivatives of $u_{\max}$?

*Remark* 2.1.6. Clearly, there hold analogous results for the minimum of the solution of the opposite partial differential inequality. Moreover, the maximum principle for elliptic equations easily follows as the special case where all the quantities around do not depend on the time variable $t$.

## 2.2 Comparison Principle

**Theorem 2.2.1 (Comparison Principle for Mean Curvature Flow).** *Let $\varphi : M_1 \times [0,T) \to \mathbb{R}^{n+1}$ and $\psi : M_2 \times [0,T) \to \mathbb{R}^{n+1}$ be two hypersurfaces moving by mean curvature, with $M_1$ compact. Then the distance between them is nondecreasing in time.*

*Proof.* The distance between the two hypersurfaces $\varphi_t : M_1 \to \mathbb{R}^{n+1}$ and $\psi_t : M_2 \to \mathbb{R}^{n+1}$ at time $t$, is given by $d^\varphi_\psi(t) = \inf_{p \in M_1, q \in M_2} |\varphi(p,t) - \psi(q,t)|$. This function is locally Lipschitz in time, as the curvature is locally bounded and the two hypersurfaces move by mean curvature, so it is differentiable almost everywhere and we assume that $t$ is a differentiability point.

This infimum is actually a minimum as $M_1$ is compact, suppose then that it is positive and let $(p_t, q_t)$ be any pair realizing such a minimum.

It is easy to see that, by minimality, the respective tangent spaces at $p_t$ and $q_t$ of the two hypersurfaces have to be parallel. Then we can write locally $\varphi(p,t)$ and $\psi(p,t)$ as graphs of two functions $f(p,t)$ and $h(p,t)$ over one of these tangent spaces for a small interval of time $(t-\varepsilon, t+\varepsilon)$. We can assume that $\langle e_1, \ldots, e_n \rangle \subset \mathbb{R}^{n+1}$ is such a tangent space with $\varphi(p_t,t) = (0, f(0,t))$ and $\psi(q_t,t) = (0, h(0,t))$ at time $t$; moreover $f(0,t) > h(0,t)$.

We know, by Exercise 1.3.8 that

$$f_t = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} \quad \text{and} \quad h_t = \Delta h - \frac{\mathrm{Hess} h(\nabla h, \nabla h)}{1 + |\nabla h|^2}.$$

Again, by minimality, the function $f(x,t)-h(x,t)$ has a minimum at $x=0$, hence, $\Delta f(0,t)-\Delta h(0,t)\geq 0$ and $\nabla f(0,t)=\nabla h(0,t)=0$, but we saw that for graphs, $\Delta f(0,t) = \mathrm{H}^{\varphi}(p_t,t)\langle \nu^{\varphi}(p_t,t)\,|\,e_{n+1}\rangle$ and $\Delta h(0,t) = \mathrm{H}^{\psi}(q_t,t)\langle \nu^{\psi}(q_t,t)\,|\,e_{n+1}\rangle$, thus,

$$\langle \mathrm{H}^{\varphi}(p_t,t)\nu^{\varphi}(p_t,t)-\mathrm{H}^{\psi}(q_t,t)\nu^{\psi}(q_t,t)\,|\,e_{n+1}\rangle = \Delta f(0,t)-\Delta h(0,t)\geq 0\,.$$

Now we have $\frac{p_t-q_t}{|p_t-q_t|}=e_{n+1}$ by construction and, by Lemma 2.1.3, we can conclude, as this analysis holds for all the pairs of points realizing the minimum, that

$$\begin{aligned}
\frac{d}{dt}d^{\varphi}_{\psi}(t) &= \inf_{(p_t,q_t)\in M_1\times M_2\ \text{with}\ |\varphi(p_t,t)-\psi(q_t,t)|=d^{\varphi}_{\psi}(t)} \frac{\partial}{\partial t}|\varphi(p_t,t)-\psi(q_t,t)|\\
&= \inf_{(p_t,q_t)\in M_1\times M_2\ \text{with}\ |\varphi(p_t,t)-\psi(q_t,t)|=d^{\varphi}_{\psi}(t)} \frac{\langle p_t-q_t\,|\,\mathrm{H}^{\varphi}\nu^{\varphi}-\mathrm{H}^{\psi}\nu^{\psi}\rangle}{|p_t-q_t|}\\
&= \inf_{(p_t,q_t)\in M_1\times M_2\ \text{with}\ |\varphi(p_t,t)-\psi(q_t,t)|=d^{\varphi}_{\psi}(t)} \langle \mathrm{H}^{\varphi}\nu^{\varphi}-\mathrm{H}^{\psi}\nu^{\psi}\,|\,e_{n+1}\rangle\\
&\geq 0\,.
\end{aligned}$$

If the minimum is zero, there is nothing to show; obviously the derivative, if it exists, cannot be negative. □

**Exercise 2.2.2.** Show the following facts for a compact hypersurface moving by mean curvature.

- The diameter of the hypersurface decreases during the flow.
- The circumradius of the hypersurface (the radius of the smallest sphere enclosing the hypersurface) decreases.

**Corollary 2.2.3.** *Let $\varphi: M_1\times[0,T)\to\mathbb{R}^{n+1}$ and $\psi: M_2\times[0,T)\to\mathbb{R}^{n+1}$ be two hypersurfaces moving by mean curvature such that $M_1$ is compact, $M_2$ is embedded and $\varphi(M_1,0)$ is strictly "inside" $\psi(M_2,0)$. Then $\varphi(M_1,t)$ remains strictly "inside" $\psi(M_2,t)$ for every time $t\in[0,T)$.*

*Proof.* This is an easy consequence of the fact that the distance between the two hypersurfaces is nondecreasing, so it cannot get to zero, as it starts positive. Hence, the hypersurface "inside" cannot "touch" the other during the flow. □

*Remark* 2.2.4. By means of the continuous dependence result in Theorem 1.5.1 one has a slight improvement of the previous corollary, allowing the two hypersurfaces, one "inside" the other, to have common points at the initial time. To prove this fact one can "push" a little inside the initial hypersurface $\varphi_0$ along the gradient of the distance function from $\psi(M_2,0)$ in a local small tubular neighborhood ($M_1$ is compact), then conclude by the above corollary and the continuous dependence of the flow on the initial hypersurface.

By means of the strong maximum principle we can actually show something more, that is, evolving by mean curvature, the distance between two connected hypersurfaces (with at least one compact) with possibly only tangent intersections and such that they "do not cross each other", is always increasing, otherwise they must coincide.

This can be seen by using again the idea of the proof of Theorem 1.5.1, writing the two hypersurfaces as graphs over the initial "external" hypersurface in a small regular tubular neighborhood of this latter and applying the strong maximum principle to the "height" functions representing them. As a preliminary step, one has to consider an "intermediate" hypersurface close enough to the "external" one which stays in its tubular neighborhood for some positive time. We leave the technical details to the reader as an exercise.

In other words, if two connected hypersurfaces (one compact "inside" the other) touch each other at time zero but they are not the same, immediately they become disjoint, at every positive time.

Even more, in the special case of curves in the plane the number of intersections (or of self-intersections) is nonincreasing in time, see [14, 16].

Applying Corollary 2.2.3 to the case that $\varphi(M_2, 0)$ is a sphere of radius $R$, we have the following estimate for the maximal time of smooth existence.

**Corollary 2.2.5.** *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface. If $\varphi(M, 0) \subset B_R(x_0)$ then the flow is contained in $B_R(x_0)$ at every time and $T \leq R^2/(2n)$.*

*Hence, the mean curvature flow of every compact immersed hypersurface develops a singularity in finite time.*

*In particular, if $T_{\max}$ is the maximal time of smooth existence of the flow, then $T_{\max} \leq \operatorname{diam}^2_{\mathbb{R}^{n+1}}[\varphi(M, 0)]/2n$.*

*Proof.* We have already seen that a sphere of radius $R$ shrinks to a point with the rule $R(t) = \sqrt{R^2 - 2nt}$, hence at time $t = R^2/(2n)$ its radius gets to zero. As $\varphi(M, t) \subset B_{\sqrt{R^2-2nt}}(x_0)$, at most at time $t = R^2/(2n)$ the evolving hypersurface $\varphi_t$ must develop a singularity, since at such time it cannot be an immersion.
The last claim is trivial. □

Another consequence of the maximum principle is the following characterization of the points of $\mathbb{R}^{n+1}$ "reached" by the flow at time $T$, that is, an estimate on the rate of convergence to a limit hypersurface as $t \to T$ (this will be particularly interesting when $T$ is a singular time). Roughly speaking, if a hypersurface moving by mean curvature is "reaching" a point of the Euclidean space at some time, then it cannot stay "too far" from such a point in the past.

**Proposition 2.2.6.** *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be a mean curvature flow and define $\mathcal{S}$ to be the set of points $x \in \mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i, t_i) \in M \times [0, T)$ with $t_i \nearrow T$ and $\varphi(p_i, t_i) \to x$.*

*Then, $\mathcal{S}$ is closed (and bounded if $M$ is compact), moreover $x \in \mathcal{S}$ if and only if for every $t \in [0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center $x$ intersects $\varphi(M,t)$.*

*Proof.* One implication is obvious.
Suppose that $x \in \mathcal{S}$ and let $d_t(x) = \min_{p\in M} |\varphi(p,t) - x|$, that is, the Euclidean distance from $x$ to the hypersurface at time $t$.
The function $d_t : [0,T) \to \mathbb{R}$ is obviously locally Lipschitz and at a differentiability time with $d_t(x) > 0$, by Hamilton's trick, Lemma 2.1.3, we have

$$d_t'(x) = \frac{\partial}{\partial t}|\varphi(q,t) - x| \geq \frac{\mathrm{H}(q,t)\langle \nu(q,t) \,|\, \varphi(q,t) - x\rangle}{|\varphi(q,t) - x|}$$

for any point $q \in M$ such that $d_t(x) = |\varphi(q,t) - x|$.
As the closed ball $\overline{B}_{d_t(x)}(x)$ intersects the hypersurface $\varphi_t$ only on its boundary and the vector $\frac{\varphi(q,t)-x}{|\varphi(q,t)-x|}$ is parallel to the normal $\nu(q,t)$ by minimality, an easy geometric argument on the principal eigenvalues of the second fundamental form shows that

$$\frac{\mathrm{H}(q,t)\langle \nu(q,t) \,|\, \varphi(q,t) - x\rangle}{|\varphi(q,t) - x|} \geq -n/d_t(x)\,.$$

Hence, we conclude that for almost every time $t \in [0,T)$,

$$d_t'(x) \geq -n/d_t(x)$$

if $d_t(x) \neq 0$.
Integrating this differential inequality on $[t,s]$ we get $d_t^2(x) - d_s^2(x) \leq 2n(s-t)$ and by the hypothesis on $x$ we have $d_{t_i}^2(x) \to 0$, hence

$$d_t^2(x) = \lim_{i\to\infty} d_t^2(x) - d_{t_i}^2(x) \leq \lim_{i\to\infty} 2n(t_i - t) = 2n(T-t)$$

which is the thesis of the proposition.
The closure of $\mathcal{S}$ is obvious, if $M$ is compact $\mathcal{S}$ is clearly also bounded by Corollary 2.2.5. □

A very important fact about hypersurfaces moving by mean curvature is the following.

**Proposition 2.2.7.** *If the initial hypersurface is compact and embedded, then it remains embedded during the flow.*

*Proof.* Given the mean curvature flow $\varphi_t$, if the hypersurface $\varphi_0$ is embedded it remains so for a small positive time, otherwise we will have a sequence of points and times, with $\varphi(p_i,t_i) = \varphi(q_i,t_i)$ and $t_i \to 0$, then, extracting a subsequence (not relabeled) such that $p_i \to p$ and $q_i \to q$, either $p \neq q$ so $\varphi(p,0) = \varphi(q,0)$, which is a contradiction, or $p = q$. By the smooth existence of the flow, in particular by the nonsingularity of the differential of $\partial_x \varphi(p,t)$ there exists a ball $B \subset M$ around $p$ such that for $t \in [0,\varepsilon)$ the map $\varphi_t|_B$ is one-to-one, which is in contradiction with the hypotheses.

This short time embeddedness property is also immediate by revisiting the proof of the short time existence theorem, representing the moving hypersurfaces as graphs on the initial one.

This argument also implies that the embeddedness holds in an open time interval, then we assume that $T > 0$ is the first time such that the hypersurface $\varphi_t$ is no more embedded. The set $S$ of pairs $(p,q)$ with $p \neq q$ and $\varphi(p,T) = \varphi(q,T)$ is a nonempty closed set disjoint from the diagonal in $M \times M$, otherwise $\varphi_T$ fails to be an immersion at some point in $M$. Then, we can find a smooth open neighborhood $\Omega$ of the diagonal with $\overline{\Omega} \cap S = \emptyset$.

We consider the quantity

$$C = \inf_{t \in [0,T]} \inf_{(p,q) \in \partial\Omega} |\varphi(p,t) - \varphi(q,t)| \,,$$

then $C$ is positive, as $\overline{\Omega} \cap S = \emptyset$ and $\partial\Omega$ is compact. We claim that the function

$$L(t) = \min_{(p,q) \in M \times M \setminus \Omega} |\varphi(p,t) - \varphi(q,t)| \,,$$

is bounded from below by $\min\{L(0), C\} > 0$ on $[0,T]$, this is clearly in contradiction with the fact that $S$ is nonempty and contained in $M \times M \setminus \Omega$.

If at some time $L(t) < C$ it follows that $L(t)$ is achieved by some pairs $(p,q)$ not belonging to $\partial\Omega$, then $(p,q)$ are inner points of $M \times M \setminus \Omega$ and a geometric argument analogous to the one of the comparison Theorem 2.2.1 shows that $\frac{dL(t)}{dt} \geq 0$, hence $L(t)$ is nondecreasing in time. This last fact clearly implies the claim. $\square$

*Remark* 2.2.8. Theorem 2.2.1 and Proposition 2.2.7 also hold if the involved hypersurfaces are not compact, with some additional assumptions on the behavior at infinity (for instance, uniform bounds on the curvature), the analysis is anyway more complicated.

## 2.3 Evolution of Curvature

Now we derive the evolution equations for $g$, $\nu$, $\Gamma^i_{jk}$, A and H. We already know that

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{H} h_{ij} \,.$$

Differentiating the formula $g_{is} g^{sj} = \delta_i^j$ we get

$$\frac{\partial}{\partial t} g^{ij} = -g^{is} \frac{\partial}{\partial t} g_{sl} g^{lj} = 2\mathrm{H} g^{is} h_{sl} g^{lj} = 2\mathrm{H} h^{ij} \,.$$

The derivative of the normal $\nu$ is given by

$$\left\langle \frac{\partial \nu}{\partial t} \,\middle|\, \frac{\partial \varphi}{\partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial t \partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial (\mathrm{H}\nu)}{\partial x_i} \right\rangle = -\frac{\partial \mathrm{H}}{\partial x_i}\,.$$

Finally the derivative of the Christoffel symbols is

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} = &\frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}g_{kl}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t}g_{jl}\right) - \frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\
&+ \frac{1}{2}\frac{\partial}{\partial t}g^{il}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right\} \\
=&\frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\
&+ \frac{1}{2}g^{il}\Big\{\frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl} + \frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} + \frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl} \\
&\qquad + \frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - \frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl} - \frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl}\Big\} \\
&- \frac{1}{2}g^{is}\frac{\partial}{\partial t}g_{sz}g^{zl}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right\} \\
=&\frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\
&+ g^{il}\frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - g^{is}\frac{\partial}{\partial t}g_{sz}\Gamma^z_{jk} \\
=&\frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\
=&-g^{il}\left\{\nabla_j(\mathrm{H}h_{kl}) + \nabla_k(\mathrm{H}h_{jl}) - \nabla_l(\mathrm{H}h_{jk})\right\} \\
=&-h^i_k\nabla_j\mathrm{H} - h^i_j\nabla_k\mathrm{H} + h_{jk}\nabla^i\mathrm{H} - \mathrm{H}(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk})\,.
\end{aligned}$$

Summarizing, we have

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= -2\mathrm{H}h_{ij} \\
\frac{\partial}{\partial t}g^{ij} &= 2\mathrm{H}h^{ij} \\
\frac{\partial}{\partial t}\nu &= -\nabla\mathrm{H} \\
\frac{\partial}{\partial t}\Gamma^i_{jk} &= \nabla\mathrm{H} * \mathrm{A} + \mathrm{H} * \nabla\mathrm{A} = \nabla\mathrm{A} * \mathrm{A}\,.
\end{aligned}$$

**Proposition 2.3.1.** *The second fundamental form satisfies the evolution equation*

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,. \tag{2.3.1}$$

*It follows that*

$$\frac{\partial}{\partial t}h_i^j = \Delta h_i^j + |\mathrm{A}|^2 h_i^j\,, \tag{2.3.2}$$

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4$$

*and*

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,. \tag{2.3.3}$$

*Proof.* Keeping in mind the Gauss–Weingarten relations (1.1.1) and the previous evolution equations, we compute

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}\left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i \partial x_j}\right\rangle\\
&= \left\langle \nu \,\middle|\, \frac{\partial^2(\mathrm{H}\nu)}{\partial x_i \partial x_j}\right\rangle - \left\langle \nabla \mathrm{H} \,\middle|\, \frac{\partial^2\varphi}{\partial x_i \partial x_j}\right\rangle\\
&= \frac{\partial^2 \mathrm{H}}{\partial x_i \partial x_j} - \mathrm{H}\left\langle \nu \,\middle|\, \frac{\partial}{\partial x_i}\left(h_{jl}g^{ls}\frac{\partial \varphi}{\partial x_s}\right)\right\rangle\\
&\quad - \left\langle \frac{\partial \mathrm{H}}{\partial x_l}\cdot\frac{\partial\varphi}{\partial x_s}g^{ls} \,\middle|\, \Gamma_{ij}^k \frac{\partial \varphi}{\partial x_k} + h_{ij}\nu\right\rangle\\
&= \frac{\partial^2 \mathrm{H}}{\partial x_i \partial x_j} - \mathrm{H}h_{jl}g^{ls}\left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i \partial x_s}\right\rangle - \Gamma_{ij}^k\frac{\partial \mathrm{H}}{\partial x_k}\\
&= \nabla_i\nabla_j \mathrm{H} - \mathrm{H}h_{il}g^{ls}h_{sj}\,.
\end{aligned}$$

Then using Simons' identity (1.1.4) we conclude that

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,.$$

The other equations follow from straightforward computations, as $\frac{\partial}{\partial t}g^{ij} = 2\mathrm{H}h^{ij}$. □

*Remark* 2.3.2. Since it will be useful in the sequel, we see in detail the evolution equations in the special one-dimensional case of the flow by curvature $\gamma : \mathbb{S}^1 \times [0, T)$ of a closed curve in the plane.

We denote by $\theta$ the parameter on $\mathbb{S}^1$ and by $s = s(\theta, t) = \int_0^\theta |\partial_\theta\gamma(\theta, t)|\,d\theta$ the arclength, $\tau = \partial_s\gamma$ is the tangent unit vector and $\nu = \mathrm{R}\tau$ is the unit normal, where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of an angle of $\pi/2$, finally $k = \langle \partial_s\tau \,|\, \nu\rangle$ is the curvature.

Notice that $\partial_s = |\gamma_\theta|^{-1}\partial_\theta$ and that the evolution equation reads $\partial_t\gamma = k\nu = \partial^2_{ss}\gamma$. Then, we easily get the commutation rule $\partial_t\partial_s = \partial_s\partial_t + k^2\partial_s$ which implies

$$\begin{aligned}
\partial_t\tau &= \partial_t\partial_s\gamma = \partial_s\partial_t\gamma + k^2\partial_s\gamma = \partial_s(k\nu) + k^2\tau = k_s\nu,\\
\partial_t\nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t\tau = -k_s\tau,\\
\partial_t k &= k_{ss} + k^3\,.
\end{aligned}$$

Now we deal with the covariant derivatives of A.

**Lemma 2.3.3.** *The following formula for the interchange of time and covariant derivative of a tensor $T$ holds:*

$$\frac{\partial}{\partial t}\nabla T = \nabla\frac{\partial}{\partial t}T + T * \mathrm{A} * \nabla\mathrm{A}\,.$$

*Proof.* We suppose that $T = T_{i_1\dots i_k}$ is a covariant tensor, the general case is analogous, as it will be clear by the following computation:

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_k} &= \frac{\partial}{\partial t}\Big(\frac{\partial T_{i_1\dots i_k}}{\partial x_j} - \sum_{s=1}^{k}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\Big)\\
&= \frac{\partial}{\partial x_j}\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}\Gamma^l_{ji_s}\frac{\partial T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}}{\partial t}\\
&\quad - \sum_{s=1}^{k}\frac{\partial}{\partial t}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\\
&= \nabla_j\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}(\mathrm{A}*\nabla\mathrm{A})^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\,,
\end{aligned}$$

which is the formula we wanted. □

**Lemma 2.3.4.** *We have, for $k > 0$, denoting by $\nabla^k$ the $k$th iterated covariant derivative,*

$$\frac{\partial}{\partial t}\nabla^k h_{ij} = \Delta\nabla^k h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}.$$

*Proof.* We work by induction on $k \in \mathbb{N}$. The case $k = 0$ is given by equation (2.3.1); we then suppose that the formula holds for $k-1$. We have, by the previous lemma,

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k h_{ij} &= \nabla\frac{\partial}{\partial t}\nabla^{k-1}h_{ij} + \nabla^{k-1}\mathrm{A}*\nabla\mathrm{A}*\mathrm{A}\\
&= \nabla\Big(\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k-1\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}\Big)\\
&\quad + \nabla^{k-1}\mathrm{A}*\nabla\mathrm{A}*\mathrm{A}\\
&= \nabla\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}.
\end{aligned}$$

Interchanging now the Laplacian and the covariant derivative and recalling that $\mathrm{Riem} = \mathrm{A}*\mathrm{A}$, we have the conclusion, as all the extra terms we get are of the form $\mathrm{A}*\mathrm{A}*\nabla^k\mathrm{A}$ and $\mathrm{A}*\nabla\mathrm{A}*\nabla^{k-1}\mathrm{A}$. □

**Proposition 2.3.5.** *The following formula holds:*

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 = \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}. \tag{2.3.4}$$

*Proof.* We compute

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 &= 2g\Big(\nabla^k \mathrm{A}, \frac{\partial}{\partial t}\nabla^k \mathrm{A}\Big) + \nabla^k \mathrm{A} * \nabla^k \mathrm{A} * \mathrm{A} * \mathrm{A} \\
&= 2g\Big(\nabla^k \mathrm{A}, \Delta\nabla^k \mathrm{A} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A}\Big) \\
&\quad + \nabla^k \mathrm{A} * \nabla^k \mathrm{A} * \mathrm{A} * \mathrm{A} \\
&= 2g\left(\nabla^k \mathrm{A}, \Delta\nabla^k \mathrm{A}\right) + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \\
&= \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,.
\end{aligned}$$

□

## 2.4 Consequences of Evolution Equations

Let us see some consequences of application of the maximum principle to evolution equations for curvature.

Suppose that we have a mean curvature flow of a compact hypersurface $M$ in the time interval $[0, T)$; we have seen that

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,.$$

First we deal with the so-called *mean convex* hypersurfaces that play a major role in the subject.

A hypersurface is mean convex if $\mathrm{H} \geq 0$ everywhere. We will see in the next proposition that this property is preserved by the mean curvature flow. Mean convexity is a significant generalization of convexity; for instance, it is general enough to allow the neckpinch behavior described in Section 1.4, in particular, mean convex hypersurfaces do not necessarily shrink to a point at the singular time.

**Proposition 2.4.1.** *Assume that the initial, compact hypersurface satisfies* $\mathrm{H} \geq 0$*. Then, under the mean curvature flow, the minimum of* $\mathrm{H}$ *is increasing, hence* $\mathrm{H}$ *is positive for every positive time.*

*Proof.* Arguing by contradiction, suppose that in an interval $(t_0, t_1) \subset \mathbb{R}^+$ we have $\mathrm{H}_{\min}(t) < 0$ and $\mathrm{H}_{\min}(t_0) = 0$ ($\mathrm{H}_{\min}$ is obviously continuous in time and $\mathrm{H}_{\min}(0) \geq 0$).

Let $|\mathrm{A}|^2 \leq C$ in such an interval. Then

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \qquad \text{implies} \qquad \frac{\partial \mathrm{H}_{\min}}{\partial t} \geq C\mathrm{H}_{\min}$$

for almost every $t \in (t_0, t_1)$.

Integrating this differential inequality in $[s, t] \subset (t_0, t_1)$ we get $\mathrm{H}_{\min}(t) \geq e^{C(t-s)}\mathrm{H}_{\min}(s)$, then sending $s \to t_0^+$ we conclude $\mathrm{H}_{\min}(t) \geq 0$ for every $t \in (t_0, t_1)$ which is a contradiction.

Since then $\mathrm{H} \geq 0$ we get

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \geq \Delta \mathrm{H} + \mathrm{H}^3/n\,.$$

With the notation of Theorem 2.1.1, we let $u = -\mathrm{H}$, $X = 0$ and $b(x) = x^3/n$, then, if $\mathrm{H}_{\min}(0) = 0$ the ODE solution $h(t)$ is always zero; so if at some positive time $\mathrm{H}_{\min}(\tau) = 0$, we have that $\mathrm{H}(\,\cdot\,, \tau)$ is constant equal to zero on $M$, but there are no compact hypersurfaces with zero mean curvature. Hence, $\mathrm{H}_{\min}$ is always increasing during the flow and $\mathrm{H}$ is positive on all $M$ at every positive time. □

Actually, this proposition can be slightly improved as follows.

**Proposition 2.4.2.** *If the initial, compact hypersurface satisfies* $|\mathrm{A}| \leq \alpha\mathrm{H}$ *for some constant* $\alpha$*, then* $|\mathrm{A}| \leq \alpha\mathrm{H}$ *for every positive time.*

*Proof.* We know that $\mathrm{H} > 0$ for every positive time, hence also $|\mathrm{A}| > 0$ for every positive time which implies that it is smooth as $|\mathrm{A}|^2$.
Let $[0, T)$ be the interval of smooth existence of the flow. Computing the evolution equation of the function $f = |\mathrm{A}| - \alpha\mathrm{H}$, we get

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{2|\mathrm{A}|}(\Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4) - \alpha(\Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2) \\
&= \Delta|\mathrm{A}| + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla \mathrm{A}|^2) + |\mathrm{A}|^3 - \alpha(\Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2) \\
&= \Delta f + |\mathrm{A}|^2 f + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla \mathrm{A}|^2) \\
&\leq \Delta f + |\mathrm{A}|^2|f|\,,
\end{aligned}$$

as the term $|\nabla|\mathrm{A}||^2 - |\nabla \mathrm{A}|^2$ is nonpositive.

Hence, choosing any $T' < T$, if $C$ is the maximum of $|\mathrm{A}|^2$ on $M \times [0, T']$, we have $\partial_t f \leq \Delta f + C|f|$ on $M \times [0, T']$. By the maximum principle Theorem 2.1.1, as $f_{\max}(0) \leq 0$, we conclude $f \leq 0$ on $M \times [0, T']$. By the arbitrariness of $T' < T$, the thesis follows. ☐

**Corollary 2.4.3.** *If $H > 0$ for the initial, compact, $n$-dimensional hypersurface, then there exists $\alpha_0 > 0$ such that $\alpha_0 |\mathrm{A}|^2 \leq \mathrm{H}^2 \leq n|\mathrm{A}|^2$ everywhere on $M$ for every time.*
*If the initial hypersurface has positive scalar curvature, then the same holds for every positive time.*

*Proof.* The first claim is immediate by the compactness of $M$ and the previous proposition (the second inequality is algebraic).
Recalling that the scalar curvature is equal to $\mathrm{H}^2 - |\mathrm{A}|^2$, positive scalar curvature implies that $\mathrm{H} > 0$ (H cannot change sign on $M$ and there is always a point where it is positive, as $M$ is compact) and $\mathrm{H}^2/|\mathrm{A}|^2 > 1$, the second part of this corollary is also a consequence of Proposition 2.4.2. ☐

**Corollary 2.4.4.** *Assume that the initial, compact hypersurface has $\mathrm{H} \geq 0$, then, if $\mathrm{A}$ is not bounded as $t \to T$ then $\mathrm{H}$ is also not bounded.*

*Proof.* Immediate consequence of Proposition 2.4.1 and the estimate of the previous corollary. ☐

Now we consider the evolution equation of $|\mathrm{A}|^2$ which implies

$$\frac{\partial}{\partial t} |\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max} \, .$$

Notice that $|\mathrm{A}|^2_{\max}$ is always positive, otherwise at some time $t$ we would have $\mathrm{A} = 0$ identically on $M$, which would imply that $M$ is a hyperplane in $\mathbb{R}^{n+1}$ in contradiction with the compactness hypothesis of $M$. Hence, we can divide both members by $|\mathrm{A}|^2_{\max}$ obtaining the following differential inequality for the locally Lipschitz function $1/|\mathrm{A}|^2_{\max}$, holding at almost every time $t \in [0, T)$,

$$-\frac{d}{dt} \frac{1}{|\mathrm{A}|^2_{\max}} \leq 2 \, .$$

Integrating in time in any interval $[t, s] \subset [0, T)$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} - \frac{1}{|\mathrm{A}(\,\cdot\,, s)|^2_{\max}} \leq 2(s - t) \, .$$

Suppose now that A is not bounded in $[0, T)$, that is, there exists a sequence of times $s_i \nearrow T$ such that $|\mathrm{A}(\,\cdot\,, s_i)|^2_{\max} \to +\infty$. Substituting these times $s_i$ in the previous inequality and sending $i \to \infty$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} \leq 2(T - t) \, .$$

**Exercise 2.4.5.** Show that the only compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are the spheres. What can be said about a compact hypersurface in $\mathbb{R}^{n+1}$ with constant $|\mathrm{A}|$?

In other words, we have proved the following.

**Proposition 2.4.6.** *If the second fundamental form* $\mathrm{A}$ *during the mean curvature flow of a compact hypersurface is not bounded as* $t \to T < +\infty$*, then it must satisfy the following lower bound for its blow-up rate:*

$$\max_{p\in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}$$

*for every* $t \in [0,T)$.
*Hence,*

$$\lim_{t\to T} \max_{p\in M} |\mathrm{A}(p,t)| = +\infty\,.$$

**Exercise 2.4.7.** Assume that the initial, compact hypersurface has $\mathrm{H} > 0$, then the maximal time of smooth existence of the flow can be estimated as $T_{\max} \leq \frac{n}{2\mathrm{H}^2_{\min}(0)}$.

**Proposition 2.4.8.** *If the second fundamental form is bounded in the interval* $[0,T)$ *with* $T < +\infty$*, then all its covariant derivatives are also bounded.*

*Proof.* By Proposition 2.3.5 we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 &= \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\\ &\leq \Delta|\nabla^k \mathrm{A}|^2 + P(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)|\nabla^k \mathrm{A}|^2 + Q(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)\,,\end{aligned}$$

where $P$ and $Q$ are smooth functions independent of time (actually they are polynomials in their arguments). Notice that in the arguments of $P, Q$ there is not $\nabla^k \mathrm{A}$; indeed, in the terms $\nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}$ there can be only one or two occurrences of $\nabla^k \mathrm{A}$, since $p + q + r = k$ and $p, q, r \in \mathbb{N}$. If there are two, suppose that $r = k$, then necessarily $p = q = 0$ and we estimate $|\mathrm{A}*\mathrm{A}*\nabla^k\mathrm{A}*\nabla^k\mathrm{A}| \leq |\mathrm{A}|^2|\nabla^k \mathrm{A}|^2$; if there is only one this means that $p, q, r < k$ and we again estimate $|\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}*\nabla^k\mathrm{A}| \leq |\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}|^2/2+|\nabla^k\mathrm{A}|^2/2$.

Reasoning by induction on $k$, being the case $k = 0$ in the hypotheses, we assume that all the covariant derivatives of $\mathrm{A}$ up to order $k-1$ are bounded, hence also $P(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)$ and $Q(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)$ are bounded, thus

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 \leq \Delta|\nabla^k \mathrm{A}|^2 + C|\nabla^k \mathrm{A}|^2 + D\,.$$

By the maximum principle, this implies

$$\frac{d}{dt}|\nabla^k \mathrm{A}|^2_{\max} \leq C|\nabla^k \mathrm{A}|^2_{\max} + D\,,$$

and since the interval $[0,T)$ is bounded, the quantity $|\nabla^k \mathrm{A}|^2_{\max}$ is also bounded, as one can obtain an easy exponential estimate for the function $u(t) = |\nabla^k \mathrm{A}|^2_{\max}$, integrating the ordinary differential inequality $u' \leq Cu + D$, holding for almost every time $t \in [0,T)$. □

**Proposition 2.4.9.** *If the second fundamental form is bounded in the interval $[0,T)$ with $T < +\infty$, then $T$ cannot be a singular time for the mean curvature flow of a compact hypersurface $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$.*

*Proof.* By the previous proposition we know that all the covariant derivatives of A are bounded by constants depending on $T$ and the geometry of the initial hypersurface. As H is bounded, we have

$$|\varphi(p,t) - \varphi(p,s)| \leq \int_s^t |\mathrm{H}(p,\xi)|\, d\xi \leq C(t-s)$$

for every $0 \leq s \leq t < T$, then the maps $\varphi_t = \varphi(\,\cdot\,,t)$ uniformly converge to a continuous limit map $\varphi_T : M \to \mathbb{R}^{n+1}$ as $t \to T$.
We fix now a vector $v = \{v^i\} \in T_pM$,

$$\frac{d}{dt} \log |v|^2_g = \frac{\frac{\partial g_{ij}}{\partial t} v^i v^j}{|v|^2_g} = \frac{-2\mathrm{H} h_{ij} v^i v^j}{|v|^2_g} \leq C \frac{|\mathrm{A}|^2 |v|^2_g}{|v|^2_g} \leq C;$$

then, for every $0 \leq s \leq t < T$,

$$\left| \log \frac{|v|^2_{g(t)}}{|v|^2_{g(s)}} \right| \leq \int_s^t \left| \frac{d}{d\xi} \log |v|^2_{g(\xi)} \right| d\xi \leq C(t-s)$$

which implies that the metrics $g(t)$ are all equivalent and the norms $|\cdot|_{g(t)}$ uniformly converge, as $t \to T$, to an equivalent norm $|\cdot|_T$ which is continuous. As the parallelogram identity passes to the limit, it must hold also for $|\cdot|_T$, hence this latter comes from a metric tensor $g_T$ which can be obtained by polarization. Moreover, since $g_T$ is equivalent to all the other metrics, it is also positive definite. Another consequence of such equivalence is that we are free to use any of these metrics in doing our estimates.

By the evolution equation for the Christoffel symbols, we see that

$$\left|\Gamma^k_{ij}(t)\right| \leq \left|\Gamma^k_{ij}(0)\right| + \int_0^t \left| \frac{\partial}{\partial \xi} \Gamma^k_{ij}(\xi) \right| d\xi \leq C + \int_0^T |\mathrm{A} * \nabla \mathrm{A}|\, d\xi \leq C + DT\,,$$

for some constants depending only on the initial hypersurface. Thus, the Christoffel symbols are equibounded in time, after fixing a local chart. This implies for every tensor $S$,

$$\left| \left| \frac{\partial S}{\partial x_i} \right| - |\nabla_i S| \right| \leq C|S|\,,$$

that is, the derivatives in coordinates differ by the relative covariant ones by equibounded terms.

*In the rest of the proof, by simplicity, we will denote by $\partial$ the coordinate derivatives and by $\nabla$ the covariant ones.*

As the time derivative of the Christoffel symbols is a tensor of the form $\mathrm{A} * \nabla\mathrm{A}$, we have

$$|\partial_t \partial^s_{l_1\dots l_s}\Gamma^k_{ij}| = |\partial^s_{l_1\dots l_s}\partial_t\Gamma^k_{ij}| = |\partial^s_{l_1\dots l_s}\mathrm{A} * \nabla\mathrm{A}|\,,$$

hence, by an induction argument on the order $s$ and integration as above, one can show that $\left|\partial^s_{l_1\dots l_s}\Gamma^k_{ij}\right| \leq C$ for every $s \in \mathbb{N}$.
Then, again by induction, the following formula (where we avoid indicating the indices) relating the iterated covariant and coordinate derivatives of a tensor $S$, holds:

$$|\,|\nabla^s S| - |\partial^s S|\,| \leq \sum_{i=1}^{s} \sum_{j_1+\dots+j_i+k\leq s-1} \left|\partial^{j_1}\Gamma\dots\partial^{j_i}\Gamma\partial^k S\right| \leq C\sum_{k=1}^{s-1}\left|\partial^k S\right|\,.$$

This implies that if a tensor has all its covariant derivatives bounded, also all the coordinate derivatives are bounded. In particular this holds for the tensor A, that is, $\left|\partial^k\mathrm{A}\right| \leq C_k$. Moreover, by induction, as $\nabla^k g = 0$ all the coordinate derivatives of the metric tensor $g$ are equibounded.

We already know that $|\varphi|$ is bounded and $|\partial\varphi| = 1$, then by the Gauss–Weingarten relations (1.1.1),

$$\partial^2\varphi = \Gamma\partial\varphi + \mathrm{A}\nu\,, \qquad \partial\nu = \mathrm{A} * \partial\varphi\,,$$

we get

$$\begin{aligned}
|\partial^k\varphi| &= \left|\sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\Gamma\partial^{i+1}\varphi + \sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\mathrm{A}\partial^i\nu\right| \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}|\partial^{i-1}(\mathrm{A} * \partial\varphi)| + C \\
&= C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\left|\sum_{p+q+r=i-1}\partial^p\mathrm{A} * \partial^q g * \partial^{r+1}\varphi\right| + C \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\sum_{r=0}^{i-1}|\partial^{r+1}\varphi| + C \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}|\partial^{i}\varphi| + C \\
&\leq C\sum_{i=0}^{k-1}|\partial^{i}\varphi|
\end{aligned}$$

where we estimated with a constant all the occurrences of $\partial^k \mathrm{A}$ and $\partial^k g$. Hence, we obtain by induction that $|\partial^k \varphi| < C_k$ for constants $C_k$ independent of time $t \in [0, T)$. By the Ascoli–Arzelà theorem we can conclude that $\varphi_T : M \to \mathbb{R}^{n+1}$ is a smooth immersion and the convergence $\varphi(\,\cdot\,, t) \to \varphi_T$ is in $C^\infty$.

Moreover, with the same argument, repeatedly differentiating the evolution equation $\partial_t \varphi = \mathrm{H}\nu$ one gets also uniform boundedness of the time derivatives of the map $\varphi$, that is $|\partial_t^s \partial_x^k \varphi| \leq C_{s,k}$. Hence the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ can be extended smoothly to the boundary of the domain of $\varphi$ with the map $\varphi_T$.

By means of the short time existence Theorem 1.5.1 we can now "restart" the flow with the immersion $\varphi_T$, obtaining a smooth extension of the map $\varphi$ which is in contradiction with the fact that $T$ was the maximal time of smooth existence. $\square$

**Open Problem 2.4.10.** Recently the condition of bounded second fundamental form was weakened by Le and Sesum [84] to a lower bound on A and an integral bound on H.
An interesting open problem is whether actually a uniform bound only on the mean curvature H is sufficient to exclude singularities during the flow (see [85]).

Thus, we conclude this section stating the following slightly improved version of Theorem 1.5.1.

**Theorem 2.4.11.** *For any initial, compact, smooth hypersurface immersed in* $\mathbb{R}^{n+1}$ *there exists a unique mean curvature flow which is smooth in a maximal time interval* $[0, T_{\max})$.
*Moreover,* $T_{\max}$ *is finite and*

$$\max_{p \in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T_{\max} - t)}}$$

*for every* $t \in [0, T_{\max})$.

Notice that it follows that the maximal time of smooth existence of the flow can be estimated from below as $T_{\max} \geq \frac{1}{2|\mathrm{A}(\,\cdot\,,0)|^2_{\max}}$.

## 2.5 Convexity Invariance

Corollary 2.4.3 is a consequence of a more general invariance property of the elementary symmetric polynomials of the curvatures, as we are going to show.
We recall that the *elementary symmetric polynomial* of degree $k$ of $\lambda_1, \dots, \lambda_n$ is defined as

$$\mathrm{S}_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

for $k = 1, \dots, n$. In particular, if $\lambda_i$ are the eigenvalues of the second fundamental form A we have $\mathrm{S}_1 = \mathrm{H}$, $\mathrm{S}_2$ is the scalar curvature and $|\mathrm{A}|^2 = \mathrm{S}_1^2 + 2\mathrm{S}_2$.

It is not difficult to show that

$$\begin{aligned}
\lambda_1 \geq 0, \dots, \lambda_n \geq 0 \quad &\Longleftrightarrow \quad \mathrm{S}_1 \geq 0, \dots, \mathrm{S}_n \geq 0\,, \\
\lambda_1 > 0, \dots, \lambda_n > 0 \quad &\Longleftrightarrow \quad \mathrm{S}_1 > 0, \dots, \mathrm{S}_n > 0\,.
\end{aligned} \tag{2.5.1}$$

These polynomials enjoy various concavity properties, see [73, 91].

**Proposition 2.5.1.** *Let $\Gamma_k \subset \mathbb{R}^n$ denote the connected component of $\{\mathrm{S}_k > 0\}$ containing the positive cone. Then $\mathrm{S}_l > 0$ in $\Gamma_k$ for all $l = 1, \dots, k$ and the quotient $\mathrm{S}_{k+1}/\mathrm{S}_k$ is concave on $\Gamma_k$.*

The above properties remain unchanged if we regard the polynomials $\mathrm{S}_k$ as functions of the Weingarten operator $h^i_j$ instead of the principal curvatures, as we have the following algebraic result, see [9, Lemma 2.22] or [73, Lemma 2.11].

**Proposition 2.5.2.** *Let $f(\lambda_1, \dots, \lambda_n)$ be a symmetric convex (concave) function of its arguments and let $F(A) = f$ (eigenvalues of $A$) for any $n \times n$ symmetric matrix $A$ whose eigenvalues belong to the domain of $f$. Then $F$ is convex (concave).*

We are now ready to derive the evolution equations of some relevant quantities and to apply the maximum principle to obtain some invariance properties.

**Proposition 2.5.3.** *Let $F(h^i_j)$ be a homogeneous function of degree one. Let $\varphi$ be a mean curvature flow of a compact, $n$-dimensional hypersurface with $\mathrm{H} > 0$ and such that $h^i_j$ belongs everywhere to the domain of $F$. Then,*

$$\frac{\partial}{\partial t}\frac{F}{\mathrm{H}} - \Delta\frac{F}{\mathrm{H}} = \frac{2}{\mathrm{H}}\Big\langle \nabla \mathrm{H} \,\Big|\, \nabla\frac{F}{\mathrm{H}}\Big\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\, g^{pq}\nabla_p h^i_j \nabla_q h^k_l\,.$$

*As a consequence, if $F$ is concave (convex), any pinching of the form $F \geq \alpha\mathrm{H}$ ($F \leq \alpha\mathrm{H}$) is preserved during the flow by the maximum principle, as the last term is then nonnegative (nonpositive).*

*Proof.* A straightforward computation using formula (2.3.2) in Proposition 2.3.1 and Euler's theorem on homogeneous functions yields

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{F}{\mathrm{H}} &= \frac{1}{H}\frac{\partial F}{\partial h^i_j}(\Delta h^i_j + |\mathrm{A}|^2 h^i_j) - \frac{F}{\mathrm{H}^2}(\Delta\mathrm{H} + |\mathrm{A}|^2\mathrm{H}) \\
&= \Delta\frac{F}{\mathrm{H}} + \frac{2}{\mathrm{H}}\Big\langle \nabla \mathrm{H} \,\Big|\, \nabla\frac{F}{\mathrm{H}}\Big\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\, g^{pq}\nabla_p h^i_j \nabla_q h^k_l\,.
\end{aligned}$$

□

In particular, the previous proposition can be applied to the quantity $F = \mathrm{S}_{k+1}/\mathrm{S}_k$, provided $\mathrm{S}_k \neq 0$. This leads to the following result, which generalizes Corollary 2.4.3.

**Proposition 2.5.4.** *Let the initial, compact hypersurface satisfy $\mathrm{S}_k > 0$ everywhere for a given $k \in \{1, \dots, n\}$ and let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be its evolution by mean curvature. Then, for any $i = 2, \dots, k$ there exists $\alpha_i$ such that $\mathrm{S}_i \geq \alpha_i \mathrm{H}^i > 0$ for every $p \in M$ and $t \in [0, T)$.*

*Proof.* We assume that the hypersurface $M$ is connected, otherwise we argue component by component.
For every pair of of points $p$ and $q$ in $M$, the set of principal curvatures at $p$ and the set of principal curvatures at $q$ belong to the same connected component of $\{\mathrm{S}_k > 0\} \subset \mathbb{R}^n$, seeing $\mathrm{S}_k$ as a map from $\mathbb{R}^n$ to $\mathbb{R}$ (connect with an arc the two points). Then, as the initial hypersurface is compact, there exists a point $p \in M$ where all the principal curvatures are positive (consider a tangent sphere containing the hypersurface), hence, the set of principal curvatures at all the points of $M$ belongs to the connected component $\Gamma_k$ of the positive cone defined in Proposition 2.5.1. Hence, for every $i = 1, \dots, k$ we have $\mathrm{S}_i > 0$ everywhere on the initial hypersurface. In particular $\mathrm{H} = \mathrm{S}_1 > 0$ and, by compactness, we have $\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1}$ for suitable constants $\beta_i > 0$, for any $i = 2, \dots, k$.
We know from Proposition 2.4.2 that $\mathrm{H} > 0$ everywhere on $M$ for every $t \in [0, T)$. Then we can consider the quotient $\mathrm{S}_2/\mathrm{H}^2 = \mathrm{S}_2/(\mathrm{H}\mathrm{S}_1)$ which is well defined for every $t$ and it is greater than $\beta_2$ at time $t = 0$. By Proposition 2.5.3 its minimum is nondecreasing, hence $\mathrm{S}_2 \geq \beta_2 \mathrm{H}^2$ for every $t \in [0, T)$.
We now apply the same procedure to the quotient $\mathrm{S}_3/(\mathrm{H}\mathrm{S}_2)$ to conclude that it is greater than $\beta_3$ for every $t \in [0, T)$, then in general $\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1}$ for $i = 2, \dots, k$. Multiplying together all these inequalities we get

$$\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1} \geq \beta_i \beta_{i-1} \mathrm{H}^2 \mathrm{S}_{i-2} \geq \cdots \geq \beta_i \beta_{i-1} \cdots \beta_2 \mathrm{H}^i$$

and the claim follows by setting $\alpha_i = \beta_i \beta_{i-1} \cdots \beta_2$. $\square$

**Corollary 2.5.5.** *If the initial, compact hypersurface is strictly convex, it remains strictly convex under the mean curvature flow.*

*Proof.* Strict convexity is equivalent to the set of conditions $\mathrm{S}_1, \dots, \mathrm{S}_n > 0$ on the eigenvalues of the second fundamental form, by relations (2.5.1) and these conditions are preserved under the mean curvature flow, by the previous proposition. $\square$

*Remark* 2.5.6. By Hamilton's strong maximum principle for tensors in [56, Section 8] (Theorem C.1.3 in Appendix C), if an initial, compact hypersurface is only convex (not necessarily strictly convex), then it becomes immediately strictly convex. Even more precisely, in this case, the smallest eigenvalue of the second fundamental form on all $M$ increases in time.
Indeed, the Weingarten operator is nonnegative definite for every positive time and satisfies (see Proposition 2.3.1)

$$\frac{\partial}{\partial t} h_i^j = \Delta h_i^j + |\mathrm{A}|^2 h_i^j \,,$$

then by Theorem C.1.3 its rank (hence the rank of A) is constant in some time interval $(0, \delta)$; moreover, the null space is invariant by parallel transport and time. Then, supposing that such rank $m$ is less than the dimension $n$ of the hypersurface,

we have an $(n-m)$-dimensional subspace $N_p \subset T_pM$ at every point $p \in M$, invariant by parallel transport, where $\mathrm{A}_p(v,v)=0$ for every $v \in N_p$.
If $v \in T_pM$ is a vector in the null space, any geodesic $\gamma$ in $M$ starting at $p$ is also a geodesic in $\mathbb{R}^{n+1}$ as $\dot\gamma$ remains always in the null space of A and

$$\nabla^{\mathbb{R}^{n+1}}_{\dot\gamma}\dot\gamma = \nabla^M_{\dot\gamma}\dot\gamma + \mathrm{A}(\dot\gamma,\dot\gamma)\nu = 0\,.$$

Hence, all the $(n-m)$-dimensional null space (as an affine subspace of $\mathbb{R}^{n+1}$) is contained in $M$, which is in contradiction with the compactness of $M$.

*Remark* 2.5.7. If the initial hypersurface is not convex, it is not true that the smallest eigenvalue of A increases; think of Angenent's homothetically shrinking torus we mentioned in Section 1.4 (see [17]).

Notice that the results about the strict monotonicity of geometric quantities during the flow are valid when the initial hypersurface is compact and can fail otherwise. For instance, an evolving cylinder does not become immediately strictly convex.

**Proposition 2.5.8.** *If for a constant $\alpha \in \mathbb{R}$ there holds $\mathrm{A} \geq \alpha \mathrm{H} g$ (as forms) for the initial, compact hypersurface, this condition is preserved during the mean curvature flow.*

*Proof.* We consider the function $f = h_{ij}v^iv^j - \alpha\mathrm{H}g_{ij}v^iv^j$ where $v^i(p,t)$ is a time-dependent smooth vector field such that $\partial v^i/\partial t = \mathrm{H}h^i_kv^k$,

$$\begin{aligned}
\frac{\partial f}{\partial t} =& \frac{\partial h_{ij}}{\partial t}v^iv^j + 2h_{ij}v^i\frac{\partial v^j}{\partial t} - \alpha\frac{\partial \mathrm{H}}{\partial t}g_{ij}v^iv^j + 2\alpha\mathrm{H}^2h_{ij}v^iv^j - 2\alpha\mathrm{H}g_{ij}v^i\frac{\partial v^j}{\partial t}\\
=&(\Delta h_{ij} - 2\mathrm{H}h^2_{ij} + |\mathrm{A}|^2h_{ij})v^iv^j + 2\mathrm{H}h^2_{ij}v^iv^j - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)g_{ij}v^iv^j\\
=&(\Delta h_{ij} + |\mathrm{A}|^2h_{ij})v^iv^j - \alpha\Delta\mathrm{H}g_{ij}v^iv^j - \alpha\mathrm{H}|\mathrm{A}|^2g_{ij}v^iv^j\\
=&\Delta(h_{ij}v^iv^j - \alpha\mathrm{H}g_{ij}v^iv^j) + |\mathrm{A}|^2(h_{ij} - \alpha\mathrm{H}g_{ij})v^iv^j\\
&- 4(\nabla_kh_{ij} - \alpha\nabla_k\mathrm{H}g_{ij})v^i\nabla^kv^j\\
&- 2(h_{ij} - \alpha\mathrm{H}g_{ij})\nabla_kv^i\nabla^kv^j - 2(h_{ij} - \alpha\mathrm{H}g_{ij})v^i\Delta v^j\\
=&\Delta f + |\mathrm{A}|^2f - 4(\nabla_kh_{ij} - \alpha\nabla_k\mathrm{H}g_{ij})v^i\nabla^kv^j\\
&- 2(h_{ij} - \alpha\mathrm{H}g_{ij})\nabla_kv^i\nabla^kv^j - 2(h_{ij} - \alpha\mathrm{H}g_{ij})v^i\Delta v^j\,.
\end{aligned}$$

Let $\mu(t)$ be the smallest value of $h_{ij}(q,t)v^iv^j - \alpha\mathrm{H}g_{ij}(q,t)v^iv^j$ for $t$ fixed, $q \in M$ and $v \in T_qM$ a unit tangent vector of $(M,g_t)$.
Since $\mu$ is a locally Lipschitz function, it is differentiable at almost every time; moreover by the hypotheses, we have $\mu(0) \geq 0$.
We suppose that there exists an open interval of time $(t_0,t_1)$ where $\mu$ is negative and $\mu(t_0)=0$. Let $t \in (t_0,t_1)$ be a differentiability point of $\mu$, then there exists a point $p \in M$ and a unit vector $v \in T_pM$ such that

$$\mu(t) = h_{ij}(p,t)v^iv^j - \alpha\mathrm{H}(p,t)g_{ij}(p,t)v^iv^j \leq h_{ij}(q,t)w^iw^j - \alpha\mathrm{H}(q,t)g_{ij}(q,t)w^iw^j$$

for every $q \in M$ and $w \in T_qM$ of unit norm. We extend the vector $v$ in space to a vector field that we still call $v$ with the following properties:

- $g_t(v(q), v(q)) \leq 1$ for every $q \in M$,
- $\nabla_{g_t} v(p) = 0$,
- $\Delta_{g_t} v(p) = 0$,

this can be done, for instance, working in local coordinates.
Now we extend $v$ also locally in time by solving the ODE $\partial v^i/\partial t = \mathrm{H}h^i_k v^k$ and we consider the associated function $f$ as above.
Notice that since $\mu(t)$ is negative in $(t_0, t_1)$, the function $f(\,\cdot\,, t)$ gets a minimum in space at $p \in M$. Indeed, if $f(q, t) < 0$, we have $v(q) \neq 0$ and

$$\begin{aligned} f(p,t) = \mu(t) &\leq \frac{h_{ij}(q,t)v^i(q)v^j(q) - \alpha\mathrm{H}(q,t)g_t(v(q),v(q))}{g_t(v(q),v(q))} \\ &= \frac{f(q,t)}{g_t(v(q),v(q))} \\ &\leq f(q,t) \end{aligned}$$

as $g_t(v(q), v(q)) \leq 1$ by construction. Hence, $\Delta f(p, t) \geq 0$ and at the point $(p, t)$ we have

$$\frac{\partial f}{\partial t} = \Delta f + |\mathrm{A}|^2 f \geq Cf$$

where $C > 0$ is a constant such that $|\mathrm{A}|^2 \leq C$ on $[0, t_1)$.
By this inequality, given $\varepsilon > 0$, there exists some $t_2 \in [t_0, t]$, such that if $\overline{t} \in [t_2, t]$ we have

$$f(p, \overline{t}) < f(p, t) - C(t - \overline{t})f(p, t) + \varepsilon(t - \overline{t})\,.$$

Since $v(p, \overline{t})$ is still a unit vector, as $\partial g(v, v)/\partial t = -2\mathrm{H}h_{ij}v^i v^j + 2g(\partial v/\partial t, v) = 0$ so the norm of $v$ is constant, we get

$$\mu(\overline{t}) \leq f(p, \overline{t}) < f(p, t) - C(t - \overline{t})f(p, t) + \varepsilon(t - \overline{t}) = \mu(t) - C(t - \overline{t})\mu(t) + \varepsilon(t - \overline{t})\,.$$

In other words $\frac{\mu(t) - \mu(\overline{t})}{t - \overline{t}} \geq C\mu(t) - \varepsilon$ and since $t$ is a differentiability time for $\mu$, passing to the limit as $\overline{t} \nearrow t$, we obtain $\mu'(t) \geq C\mu(t) - \varepsilon$.
Finally, as $\varepsilon$ is arbitrarily small, we conclude that $\mu'(t) \geq C\mu(t)$.
Since this relation holds at every differentiability time in $(t_0, t_1)$ where $\mu < 0$, hence almost everywhere, we can integrate it in the interval $(t_0, t_1)$. Recalling that $\mu(t_0) = 0$ by continuity, we conclude that $\mu(t)$ must be identically zero in $[t_0, t_1)$ which is in contradiction with the hypotheses.
Notice the similarities with the proofs of Lemma 2.1.3 and Proposition 2.4.1. □

**Exercise 2.5.9.** Show that for an initial hypersurface with $\mathrm{H} > 0$ the smallest eigenvalue of the form $h_{ij}/\mathrm{H}$ is nondecreasing during the flow.

Finally, further invariance properties for the mean curvature flow can be obtained again by means of Hamilton's maximum principle for tensors [56, Sections 4 and 8] (whose proof is a generalization of the argument above), see Appendix C. Let us first recall a definition; we say that an immersed hypersurface is $k$-convex, for some $1 \leq k \leq n$, if the sum of the $k$ smallest principal curvatures is nonnegative at every point. In particular, one-convexity coincides with convexity, while $n$-convexity means nonnegativity of the mean curvature $\mathrm{H}$, that is, mean convexity. Then we mention the following result generalizing Corollary 2.5.5 (see [75]).

**Proposition 2.5.10.** *If an initial, compact hypersurface is $k$-convex, then it is so for every positive time under the mean curvature flow.*

*Proof.* The result follows from Hamilton's maximum principle for tensors, provided we show that the inequality $\lambda_1 + \cdots + \lambda_k \geq \alpha\mathrm{H}$ describes a convex cone in the set of all matrices, and that this cone is invariant under the system of ODE's $dh^i_j/dt = |\mathrm{A}|^2 h^i_j$ for the Weingarten operator.
As

$$(\lambda_1 + \cdots + \lambda_k)(p) = \min_{\substack{e_1,\dots,e_k \in T_pM \\ g_p(e_i,e_j)=\delta_{ij}}} \left\{\mathrm{A}_p(e_1, e_1) + \cdots + \mathrm{A}_p(e_k, e_k)\right\},$$

the quantity $\lambda_1 + \cdots + \lambda_k$ is a concave function of the Weingarten operator, being the infimum of a family of linear maps. Therefore the inequality $\lambda_1+\cdots+\lambda_k \geq \alpha\mathrm{H}$ describes a convex cone of matrices. In addition, the system $dh^i_j/dt = |\mathrm{A}|^2 h^i_j$ changes the Weingarten operator by homotheties, thus leaves any cone invariant. The conclusion follows. $\square$

# Chapter 3

# Monotonicity Formula and Type I Singularities

Throughout this chapter, $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is the mean curvature flow of an $n$-dimensional, compact hypersurface in the maximal interval of smooth existence $[0,T)$.

As before we will use the notation $\varphi_t = \varphi(\,\cdot\,,t)$ and $\widetilde{\mathcal{H}}^n$ will be the $n$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ *counting multiplicities.*

## 3.1 The Monotonicity Formula for Mean Curvature Flow

We show the fundamental monotonicity formula for mean curvature flow, discovered by Huisken in [67] and then generalized by Hamilton in [58, 59].

**Lemma 3.1.1.** *Let $f : \mathbb{R}^{n+1} \times I \to \mathbb{R}$ be a smooth function. By a little abuse of notation, we denote by $\int_M f\,d\mu_t$ the integral $\int_M f(\varphi(p,t),t)\,d\mu_t(p)$. Then the following formula holds:*

$$\frac{d}{dt}\int_M f\,d\mu_t = \int_M (f_t - \mathrm{H}^2 f + \mathrm{H}\langle \nabla f \,|\, \nu\rangle)\,d\mu_t\,.$$

*Proof.* Straightforward computation. $\square$

If $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ is a solution of the backward heat equation in $\mathbb{R}^{n+1}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M u\,d\mu_t &= \int_M (u_t - \mathrm{H}^2 u + \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\,d\mu_t \\ &= -\int_M (\Delta^{\mathbb{R}^{n+1}} u + \mathrm{H}^2 u - \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\,d\mu_t\,.\end{aligned} \tag{3.1.1}$$

C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progress in Mathematics 290,
DOI 10.1007/978-3-0348-0145-4_3, © Springer Basel AG 2011

**Lemma 3.1.2.** *If $\psi : M \to \mathbb{R}^{n+1}$ is a smooth isometric immersion of an $n$-dimensional Riemannian manifold $(M, g)$, for every smooth function $u$ defined in a neighborhood of $\psi(M)$ we have*

$$\Delta_g(u(\psi)) = (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle \nu \,|\, (\nabla u)(\psi)\rangle\,,$$

*where $(\nabla^2_{\nu\nu} u)(\psi(p))$ is the second derivative of $u$ in the normal direction $\nu(p) \in \mathbb{R}^{n+1}$ at the point $\psi(p)$.*

*Proof.* Let $p \in M$ and choose normal coordinates at $p$. Set $\widetilde{u} = u \circ \psi$, then

$$\begin{aligned}
\Delta_g \widetilde{u} &= \nabla^2_{ii}(u \circ \psi)\\
&= \nabla_i \left( \frac{\partial u}{\partial y_\alpha} \frac{\partial \psi^\alpha}{\partial x_i} \right)\\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \frac{\partial \psi^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha} \frac{\partial^2 \psi^\alpha}{\partial x_i^2}\\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \frac{\partial \psi^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha} h_{ii} \nu^\alpha\\
&= (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle \nu \,|\, (\nabla u)(\psi)\rangle\,,
\end{aligned}$$

where we used the Gauss–Weingarten relations (1.1.1). □

It follows that, substituting $\Delta^{\mathbb{R}^{n+1}} u$ in formula (3.1.1) and using the previous lemma, if the function $u$ is positive we get

$$\begin{aligned}
\frac{d}{dt} \int_M u \, d\mu_t &= -\int_M (\Delta_{g(t)}(u(\varphi_t)) + \nabla^2_{\nu\nu} u + \mathrm{H}^2 u - 2\mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t\\
&= -\int_M (\nabla^2_{\nu\nu} u + \mathrm{H}^2 u - 2\mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t\\
&= -\int_M \left| \mathrm{H} - \frac{\langle \nabla u \,|\, \nu\rangle}{u} \right|^2 u \, d\mu_t + \int_M \left( \frac{|\nabla^\perp u|^2}{u} - \nabla^2_{\nu\nu} u \right) d\mu_t\,,
\end{aligned}$$

where $\nabla^\perp u$ denotes the projection on the normal space to $M$ of the gradient of $u$. Then, assuming that $u : \mathbb{R}^{n+1} \times [0, \tau) \to \mathbb{R}$ is a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ for some $\tau > 0$, we easily get the following formula:

$$\begin{aligned}
\frac{d}{dt} \left[ \sqrt{4\pi(\tau - t)} \int_M u \, d\mu_t \right] = &- \sqrt{4\pi(\tau - t)} \int_M |\mathrm{H} - \langle \nabla \log u \,|\, \nu\rangle|^2 u \, d\mu_t \qquad (3.1.2)\\
&- \sqrt{4\pi(\tau - t)} \int_M \left( \nabla^2_{\nu\nu} u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau - t)} \right) d\mu_t
\end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.
As we can see, the right-hand side consists of a nonpositive quantity and a term which is nonpositive if $\frac{\nabla^2_{\nu\nu} u}{u} - \frac{|\nabla^\perp u|^2}{u^2} + \frac{1}{2(\tau - t)} = \nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)}$ is nonnegative.

Setting $v(x,s) = u(x,\tau - s)$, the function $v : \mathbb{R}^{n+1} \times (0,\tau] \to \mathbb{R}$ is a positive solution of the standard *forward* heat equation in all $\mathbb{R}^{n+1}$ and, setting $t = \tau - s$, we have $\nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau-t)} = \nabla^2_{\nu\nu} \log v + \frac{1}{2s}$. This last expression is exactly the Li–Yau–Hamilton 2-form $\nabla^2 \log v + g/(2s)$ for positive solutions of the heat equation on a compact manifold $(N,g)$, evaluated on $\nu \otimes \nu$ (see [58]).

In the paper [58] (see also [93]) Hamilton generalized the Li–Yau differential Harnack inequality in [86] (concerning the nonnegativity of $\Delta \log v + \frac{\dim N}{2s}$) showing that, under the assumptions that $(N,g)$ has parallel Ricci tensor ($\nabla \mathrm{Ric} = 0$) and nonnegative sectional curvatures, the 2-form $\nabla^2 \log v + g/(2s)$ is nonnegative definite (Hamilton's matrix Li–Yau–Harnack inequality). In particular, in $\mathbb{R}^{n+1}$ equipped with the canonical flat metric such hypotheses clearly hold and $\nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau-t)} = \Big(\nabla^2 \log v + g^{\mathbb{R}^{n+1}}_{\mathrm{can}}/(2s)\Big)(\nu \otimes \nu) \geq 0$. Hence, assuming the boundedness in space of $v$ (equivalently of $u$), the monotonicity formula implies that $\sqrt{4\pi(\tau - t)} \int_M u \, d\mu_t$ is nonincreasing in time. We resume this discussion in the following theorem by Hamilton [58, 59].

**Theorem 3.1.3 (Huisken's Monotonicity Formula – Hamilton's Extension in $\mathbb{R}^{n+1}$).** *Assume that for some $\tau > 0$ we have a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ in $\mathbb{R}^{n+1} \times [0,\tau)$, bounded in space for every fixed $t \in [0,\tau)$; then*

$$\frac{d}{dt}\left[\sqrt{4\pi(\tau-t)} \int_M u \, d\mu_t\right] \leq -\sqrt{4\pi(\tau-t)} \int_M |\mathrm{H} - \langle \nabla \log u \,|\, \nu \rangle|^2 u \, d\mu_t$$

*in the time interval $[0, \min\{\tau, T\})$.*

*Remark* 3.1.4. In the original paper of Hamilton, the compactness of the ambient space is required (the proof is based on the maximum principle); in order to extend his result to $\mathbb{R}^{n+1}$ we assumed the boundedness in space of $u$, see Appendix D for details.

Choosing in particular a *backward* heat kernel of $\mathbb{R}^{n+1}$, that is

$$u(x,t) = \rho_{x_0,\tau}(x,t) = \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}$$

in formula (3.1.2), we get the standard Huisken's monotonicity formula, as the Li–Yau–Hamilton expression $\nabla^2_{\nu\nu} u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}$ is identically zero in this case.

**Theorem 3.1.5 (Huisken's Monotonicity Formula).** *For every $x_0 \in \mathbb{R}^{n+1}$ and $\tau > 0$ we have (see [67])*

$$\frac{d}{dt} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} \, d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_0 \,|\, \nu \rangle}{2(\tau-t)} \right|^2 d\mu_t$$

*in the time interval $[0, \min\{\tau, T\})$.*

*Hence, the integral*

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t$$

*is nonincreasing during the flow in* $[0, \min\{\tau, T\})$.

**Exercise 3.1.6.** Show that for every $x_0 \in \mathbb{R}^{n+1}$, $\tau > 0$ and a smooth function $v : M \times [0,T) \to \mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,v\,d\mu_t = &-\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x-x_0\,|\,\nu\rangle}{2(\tau-t)}\right|^2 v\,d\mu_t\\ &+\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,(v_t - \Delta_{g(t)}v)\,d\mu_t\,,\end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.
In particular if $v : M \times [0,T) \to \mathbb{R}$ is a smooth solution of $v_t = \Delta_{g(t)}v$,

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,v\,d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x-x_0\,|\,\nu\rangle}{2(\tau-t)}\right|^2 v\,d\mu_t$$

in $[0, \min\{\tau, T\})$.

## 3.2 Type I Singularities and the Rescaling Procedure

In the previous chapter we showed that the curvature must blow up at the maximal time $T$ with the lower bound

$$\max_{p\in M}|\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}\,.$$

**Definition 3.2.1.** Let $T$ be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that we have the upper bound

$$\max_{p\in M}|\mathrm{A}(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}\,,$$

we say that the flow is developing at time $T$ a *type I singularity.*
If such a constant does not exist, that is

$$\limsup_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)|\sqrt{T-t} = +\infty,$$

we say that we have a *type II singularity.*

In this chapter we deal exclusively with type I singularities and the monotonicity formula will be the main tool for the analysis. The next chapter will be devoted to type II singularities.

From now on, we assume that there exists some constant $C_0 > 1$ such that

$$\frac{1}{\sqrt{2(T-t)}} \leq \max_{p\in M} |\mathrm{A}(p,t)| \leq \frac{C_0}{\sqrt{2(T-t)}}\,, \tag{3.2.1}$$

for every $t \in [0,T)$.

Let $p \in M$ and $0 \leq t \leq s < T$, then

$$\begin{aligned}|\varphi(p,s) - \varphi(p,t)| &= \left|\int_t^s \frac{\partial\varphi(p,\xi)}{\partial t}\,d\xi\right| \leq \int_t^s |\mathrm{H}(p,\xi)|\,d\xi \\ &\leq \int_t^s \frac{C_0\sqrt{n}}{\sqrt{2(T-\xi)}}\,d\xi \leq C_0\sqrt{n(T-t)}\end{aligned}$$

which implies that the sequence of functions $\varphi(\,\cdot\,,t)$ converges as $t \to T$ to some function $\varphi_T : M \to \mathbb{R}^{n+1}$. Moreover, as the constant $C_0$ is independent of $p \in M$, such convergence is uniform and the limit function $\varphi_T$ is continuous. Finally, passing to the limit in the above inequality, we get

$$|\varphi(p,t) - \varphi_T(p)| \leq C_0\sqrt{n(T-t)}\,. \tag{3.2.2}$$

Throughout the chapter we will denote $\varphi_T(p)$ by $\widehat{p}$.

**Definition 3.2.2.** Let $\mathcal{S}$ be the set of points $x \in \mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i,t_i) \in M \times [0,T)$ with $t_i \nearrow T$ and $\varphi(p_i,t_i) \to x$.
We call $\mathcal{S}$ the set of *reachable* points.

We have seen in Proposition 2.2.6 that $\mathcal{S}$ is compact and that $x \in \mathcal{S}$ if and only if, for every $t \in [0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center $x$ intersects $\varphi(M,t)$. We show now that $\mathcal{S} = \{\widehat{p}\,|\,p \in M\}$.
Clearly $\{\widehat{p}\,|\,p \in M\} \subset \mathcal{S}$, suppose that $x \in \mathcal{S}$ and $\varphi(p_i,t_i) \to x$, then, by inequality (3.2.2) we have $|\varphi(p_i,t_i) - \widehat{p_i}| \leq C_0\sqrt{n(T-t_i)}$, hence, $\widehat{p_i} \to x$ as $i \to \infty$. As the set $\{\widehat{p}\,|\,p \in M\}$ is closed it follows that it must contain the point $x$.

We define now a tool which will be fundamental in the sequel.

**Definition 3.2.3.** For every $p \in M$, we define the *heat density* function

$$\theta(p,t) = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

and the *limit heat density* function as

$$\Theta(p) = \lim_{t\to T} \theta(p,t)\,.$$

As $M$ is compact, we can also define the *maximal heat density*

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t \tag{3.2.3}$$

and its limit $\Sigma = \lim_{t \to T} \sigma(t)$.

Clearly, $\theta(p,t) \leq \sigma(t)$, for every $p \in M$ and $t \in [0,T)$ and $\Theta(p) \leq \Sigma$ for every $p \in M$.

The function $\Theta$ is well defined as the limit exists and is finite, since $\theta(p,t)$ is monotone nonincreasing in $t$ and positive. Moreover, the functions $\theta(\,\cdot\,,t)$ are all continuous and monotonically converging to $\Theta$, hence this latter is upper semicontinuous and nonnegative.

The function $\sigma : [0,T) \to \mathbb{R}$ is also positive and monotone nonincreasing, being the maximum of a family of nonincreasing smooth functions, hence the limit $\Sigma$ is well defined and finite. Moreover, such a family is uniformly locally Lipschitz (look at the right-hand side of the monotonicity formula), hence also $\sigma$ is locally Lipschitz; then by Hamilton's trick 2.1.3, at every differentiability time $t \in [0,T)$ of $\sigma$ we have

$$\sigma'(t) = -\int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_t \,|\, \nu \rangle}{2(T-t)} \right|^2 d\mu_t \tag{3.2.4}$$

where $x_t \in \mathbb{R}^{n+1}$ is any point where the maximum defining $\sigma(t)$ is attained, that is

$$\sigma(t) = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t \,.$$

*Remark* 3.2.4. Notice that we did not define $\sigma(t)$ as the maximum of $\theta(\,\cdot\,,t)$,

$$\max_{p \in M} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t,$$

which is *taken among* $p \in M$. Clearly, this latter can be smaller than $\sigma(t)$.

We rescale now the moving hypersurfaces around $\widehat{p} = \lim_{t \to T} \varphi(p,t)$, following Huisken [67],

$$\widetilde{\varphi}(q,s) = \frac{\varphi(q,t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

and compute the evolution equation for $\widetilde{\varphi}(q,s)$ in the time interval $\left[-\frac{1}{2}\log T, +\infty\right)$,

$$\begin{aligned}\frac{\partial\widetilde{\varphi}(q,s)}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1}\frac{\partial}{\partial t}\left(\frac{\varphi(q,t)-\widehat{p}}{\sqrt{2(T-t)}}\right)\\ &= \sqrt{2(T-t)}\,\frac{\partial\varphi(q,t)}{\partial t} + \frac{\varphi(q,t)-\widehat{p}}{\sqrt{2(T-t)}}\\ &= \sqrt{2(T-t)}\,\mathrm{H}(q,t)\nu(q,t) + \widetilde{\varphi}(q,s)\\ &= \widetilde{\mathrm{H}}(q,s)\widetilde{\nu}(q,s) + \widetilde{\varphi}(q,s)\,,\end{aligned}$$

where $\widetilde{\mathrm{H}}$ is the mean curvature of the rescaled hypersurfaces $\widetilde{\varphi}_s = \widetilde{\varphi}(\,\cdot\,,s)$.

As $|\widetilde{\mathrm{A}}| = \sqrt{2(T-t)}\,|\mathrm{A}| \leq C_0 < +\infty$, all the hypersurfaces $\widetilde{\varphi}_s$ have equibounded curvatures, moreover,

$$|\widetilde{\varphi}(p,s)| = \left|\frac{\varphi(p,t(s))-\widehat{p}}{\sqrt{2(T-t(s))}}\right| \leq \frac{C_0\sqrt{2n(T-t(s))}}{\sqrt{2(T-t(s))}} = C_0\sqrt{n}$$

which implies that at every time $s$ the open ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$ intersects the hypersurface $\widetilde{\varphi}(\,\cdot\,,s)$. More precisely, the point $\widetilde{\varphi}(p,s)$ belongs to the interior of such a ball.

Then, we rescale also the monotonicity formula in order to get information on these hypersurfaces.

**Proposition 3.2.5 (Rescaled Monotonicity Formula).** *We have*

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \leq 0 \tag{3.2.5}$$

*which integrated becomes*

$$\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{s_1} - \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{s_2} = \int_{s_1}^{s_2}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,ds\,.$$

*In particular,*

$$\int_{-\frac{1}{2}\log T}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,ds \leq \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T} \leq C < +\infty\,,$$

*for a uniform constant* $C = C(\mathrm{Area}(\varphi_0), T)$ *independent of* $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ *and* $p \in M$.

*Proof.* Keeping in mind that $y = \frac{x-\widehat{p}}{\sqrt{2(T-t)}}$ and $s = -\frac{1}{2}\log(T-t)$, we have

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1}\frac{d}{dt}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s\\
&= 2(T-t)\frac{d}{dt}\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\,d\mu_t\\
&= -2(T-t)\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x-\widehat{p}\,|\,\nu\rangle}{2(T-t)}\right|^2 d\mu_t\\
&= -2(T-t)\int_M e^{-\frac{|y|^2}{2}}\left|\frac{\widetilde{\mathrm{H}}}{\sqrt{2(T-t)}} + \frac{\langle y\,|\,\widetilde{\nu}\rangle}{\sqrt{2(T-t)}}\right|^2 d\widetilde{\mu}_s\\
&= -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,.
\end{aligned}$$

The other two statements trivially follow. □

As a first consequence, we work out an upper estimate on the volume of the rescaled hypersurfaces in the balls of $\mathbb{R}^{n+1}$.
Fix a radius $R > 0$, if $B_R = B_R(0) \subset \mathbb{R}^{n+1}$, then we have

$$\begin{aligned}
\widetilde{\mathcal{H}}^n(\widetilde{\varphi}(M,s)\cap B_R) &= \int_M \chi_{B_R}(y)\,d\widetilde{\mu}_s \qquad (3.2.6)\\
&\leq \int_M \chi_{B_R}(y)e^{\frac{R^2-|y|^2}{2}}\,d\widetilde{\mu}_s\\
&\leq e^{R^2/2}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s\\
&\leq e^{R^2/2}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T}\\
&\leq \widehat{C}e^{R^2/2}
\end{aligned}$$

where the constant $\widehat{C}$ is independent of $R$ and $s$.

*Remark* 3.2.6. As

$$\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T} = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4T}}}{(2T)^{n/2}}\,d\mu_0 \leq \frac{\mathrm{Area}(\varphi_0)}{(2T)^{n/2}}\,,$$

we can choose the constant $\widehat{C}$ to be independent also of $p \in M$.

Another consequence is the following key technical lemma which is necessary in order to take the limits of integrals of functions on the sequences of rescaled hypersurfaces.

**Lemma 3.2.7 (Stone [115]).** *The following estimates hold.*

1. *There is a uniform constant* $C = C(n, \mathrm{Area}(\varphi_0), T)$ *such that, for any* $p \in M$ *and for all* $s$,
$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq C\,.$$

2. *For any* $\varepsilon > 0$ *there is a uniform radius* $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ *such that, for any* $p \in M$ *and for all* $s$,
$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq \varepsilon\,,$$
*that is, the family of measures* $e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}_s(M)$ *is* tight (*see* [32]).

*Proof.* By the rescaled monotonicity formula (3.2.5) we have that, for any $p \in M$ and for all $s$,
$$\int_M e^{-|y|^2/2}\, d\widetilde{\mu}_s \leq \int_M e^{-|y|^2/2}\, d\widetilde{\mu}_{-\frac{1}{2}\log T}\,.$$
According to Remark 3.2.6, the right-hand integral may be estimated by a constant depending only on $T$ and $\mathrm{Area}(\varphi_0)$, not on $p \in M$. Hence, we have the following estimates for all $p \in M$ and for all $s$,
$$\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1 \tag{3.2.7}$$
and
$$\int_{\widetilde{\varphi}_s(M)\cap B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_2 \tag{3.2.8}$$
where $C_1$ and $C_2$ are constants depending only on $n$, $T$ and $\mathrm{Area}(\varphi_0)$.
By the evolution equation for the rescaled hypersurfaces $\widetilde{\varphi}_s$ we get $\frac{d}{ds}\widetilde{\mu}_s = (n - \widetilde{\mathrm{H}}^2)\widetilde{\mu}_s$, then we compute for any $p$ and $s$,
$$\begin{aligned}
\frac{d}{ds}\int_M e^{-|y|}\, d\widetilde{\mu}_s &= \int_M \Big\{ n - \widetilde{\mathrm{H}}^2 - \frac{1}{|y|}\langle y \,|\, \widetilde{\mathrm{H}}\widetilde{\nu} + y\rangle \Big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&\leq \int_M \big\{ n - \widetilde{\mathrm{H}}^2 - |y| + |\widetilde{\mathrm{H}}| \big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&< \int_M \big\{ n + 1 - |y| \big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&\leq (n+1)\left\{ \int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n - \int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \right\}.
\end{aligned}$$
But then, by inequality (3.2.7) we see that we must have either
$$\frac{d}{ds}\int_M e^{-|x|}\, d\widetilde{\mu}_s < 0\,, \quad \text{or} \quad \int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1\,.$$

Hence, in view of inequality (3.2.8), it follows that either

$$\frac{d}{ds}\int_M e^{-|y|}\,d\widetilde{\mu}_s < 0\,, \quad \text{or} \quad \int_M e^{-|y|}\,d\widetilde{\mu}_s \leq C_1 + C_2\,,$$

which implies

$$\int_M e^{-|y|}\,d\widetilde{\mu}_s \leq \max\left\{(C_1+C_2), \int_M e^{-|y|}\,d\widetilde{\mu}_{-\frac{1}{2}\log T}\right\} = C_3$$

for any $p$ and $s$.
The proof of part (1) of the lemma follows by noticing that the integral quantity on the right-hand side can clearly be estimated by a constant depending on $T$ and $\mathrm{Area}(\varphi_0)$ but not on $p \in M$.

Let now again $p \in M$ and $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ arbitrary. Now subdivide $\widetilde{\varphi}_s(M)$ into "annular pieces" $\big\{\widetilde{M}_s^k\big\}_{k=0}^{\infty}$ by setting

$$\widetilde{M}_s^0 = \widetilde{\varphi}_s(M) \cap B_1(0)\,,$$

and for each $k \geq 1$,

$$\widetilde{M}_s^k = \big\{y \in \widetilde{\varphi}_s(M) \,|\, 2^{k-1} \leq |y| < 2^k\big\}\,.$$

Then, by part (1) of the lemma $\widetilde{\mathcal{H}}^n(\widetilde{M}_s^k) \leq C_3 e^{(2^k)}$ for each $k$, independently of the choice of $p$ and $s$. Hence in turn, for each $k$ we have

$$\int_{\widetilde{M}_s^k} e^{-|y|^2/2}\,d\widetilde{\mathcal{H}}^n \leq C_3 e^{-\frac{1}{2}(2^{k-1})^2} e^{(2^k)} = C_3 e^{(2^k - 2^{2k-3})}$$

again independently of the choice of $p$ and $s$.
For any $\varepsilon > 0$ we can find a $k_0 = k_0(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ such that

$$\sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon\,,$$

then, if $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ is simply taken to be equal to $2^{k_0-1}$, we have

$$\begin{aligned}\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\,d\widetilde{\mathcal{H}}^n &= \sum_{k=k_0}^{\infty}\int_{\widetilde{M}_s^k} e^{-|y|^2/2}\,d\widetilde{\mathcal{H}}^n \\ &\leq \sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon\end{aligned}$$

and we are done also with part (2) of the lemma. □

**Corollary 3.2.8.** *If a sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$ locally smoothly converges (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$, we have*

$$\int_{\widetilde{M}_\infty} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C$$

*and*

$$\lim_{i\to\infty} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{s_i} = \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n\,,$$

*where the constant $C$ is the same as in the previous lemma.*

*Proof.* Actually, it is sufficient to show only that the measures $\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}(M, s_i)$ associated to the hypersurfaces weakly$^*$-converge to the measure $\widetilde{\mathcal{H}}^n \llcorner \widetilde{M}_\infty$. Indeed, for every $R > 0$ we have,

$$\begin{aligned}\int_{\widetilde{M}_\infty \cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n &\leq \liminf_{i\to\infty} \int_{\widetilde{\varphi}(M,s_i)\cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n\\ &\leq \liminf_{i\to\infty} \int_M e^{-|y|}\, d\widetilde{\mu}_{s_i} \leq C\end{aligned}$$

by the first part of the lemma above. Sending $R$ to $+\infty$, the first inequality follows. The second statement is an easy consequence of the estimates in the second part of the lemma. □

Now we want to estimate the covariant derivatives of the rescaled hypersurfaces.

**Proposition 3.2.9 (Huisken [67]).** *For every $k \in \mathbb{N}$ there exists a constant $C_k$ depending only on $k$, $n$, $C_0$ (the constant in formula (3.2.1)) and the initial hypersurface such that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$ for every $p \in M$ and $s \in \Big[-\frac{1}{2}\log T, +\infty\Big)$.*

*Proof.* By Proposition 2.3.5 we have for the original flow,

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 = \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,,$$

hence, with a straightforward computation, noticing that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} = |\nabla^k \mathrm{A}|^2_g [2(T - t)]^{k+1}$ we get

$$\begin{aligned}\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \leq &- 2(k+1)|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ C(n,k) \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} |\widetilde{\nabla}^p \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^q \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^r \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}}.\end{aligned}$$

As $|\widetilde{\mathrm{A}}|_{\widetilde{g}}$ is bounded by the constant $C_0$, supposing by induction that for $i = 0, \dots, k-1$ we have uniform bounds on $|\widetilde{\nabla}^i \widetilde{\mathrm{A}}|_{\widetilde{g}}$ with constants $C_i = C_i(n, C_0, \varphi_0)$, we can conclude by means of Peter–Paul inequality

$$\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \leq \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k$$

for some constants $B_k$ and $D_k$ depending only on $n$, $k$, $C_0$ and the initial hypersurface.

Then,

$$\begin{aligned}
\frac{\partial}{\partial s}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \leq\, & \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\
& + B_k\widetilde{\Delta}|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_kB_{k-1}|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\
& - 2B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k + B_kD_{k-1} \\
\leq\, & \widetilde{\Delta}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\
& + B_kB_{k-1}|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k + B_kD_{k-1} \\
\leq\, & \widetilde{\Delta}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\
& + B_kB_{k-1}C^2_{k-1} + D_k + B_kD_{k-1} \\
\leq\, & \widetilde{\Delta}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \\
& - B_k(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \\
& + B_kB_{k-1}C^2_{k-1} + D_k + B_kD_{k-1} + B_k^2C^2_{k-1}
\end{aligned}$$

where we used the inductive hypothesis $|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_{k-1}$.

By the maximum principle, the function $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}$ is then uniformly bounded in space and time by a constant $C_k^2$ depending on $n$, $k$, $C_0$ and the initial hypersurface, hence $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$. By the inductive hypothesis, the thesis of the proposition follows. □

We are now ready to study the convergence of the rescaled hypersurfaces as $s \to +\infty$.

**Proposition 3.2.10.** *For every point $p \in M$ and every sequence of times $s_i \to +\infty$ there exists a subsequence (not relabelled) of times such that the hypersurfaces $\widetilde{\varphi}_{s_i}$, rescaled around $\widehat{p}$, locally smoothly converge (up to reparametrization) to some nonempty, smooth, complete limit hypersurface $\widetilde{M}_\infty$ such that $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$ for every $y \in \widetilde{M}_\infty$.*

*Any limit hypersurface satisfies $\widetilde{\mathcal{H}}^n(\widetilde{M}_\infty \cap B_R) \leq C_R$ for every ball of radius $R$ in $\mathbb{R}^{n+1}$ and for every $k \in \mathbb{N}$ there are constants $C_k$ such that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$.*

*Moreover, if the initial hypersurface was embedded, $\widetilde{M}_\infty$ is embedded.*

*Proof.* We give a sketch of the proof, following Huisken [67].

By estimate (3.2.6) there is a uniform upper bound on $\mathcal{H}^n(\widetilde{\varphi}(M,s)\cap B_R)$ for each $R$, independent of $s$. Moreover, by the uniform control on the norm of the second fundamental form of the rescaled hypersurfaces in Proposition 3.2.9, there is a number $r_0 > 0$ such that, for each $s$ and each $q \in M$, if $U^s_{r_0,q}$ is the connected component of $\widetilde{\varphi}_s^{-1}(B_{r_0}(\widetilde{\varphi}_s(q)))$ in $M$ containing $q$, then $\widetilde{\varphi}_s(U^s_{r_0,q})$ can be written as a graph of a smooth function $f$ over a subset of the ball of radius $r_0$ in the tangent hyperplane to $\widetilde{\varphi}_s(M) \subset \mathbb{R}^{n+1}$ at the point $\widetilde{\varphi}_s(q)$.

The estimates of Proposition 3.2.9 then imply that all the derivatives of such function $f$ up to the order $\alpha \in \mathbb{N}$ are bounded by constants $C_\alpha$ independent of $s$.

Following now the method in [83] we can see that, for each $R > 0$, a subsequence of the hypersurfaces $\widetilde{\varphi}(M,s)\cap B_R(0)$ must converge smoothly to a limit hypersurface in $B_R(0)$. Then, the existence of a smooth, complete limit hypersurface $\widetilde{M}_\infty$ follows from a diagonal argument, letting $R \to +\infty$. Recalling the fact that every rescaled hypersurface intersects the ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$, this limit cannot be empty. The estimates on the volume and derivatives of the curvature follow from the analogous properties for the converging sequence.

The fact that $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$ is a consequence of the rescaled monotonicity formula and of the uniform estimates on the curvature and its covariant derivatives for the rescaled hypersurfaces in Proposition 3.2.9. Indeed, by means of equation (2.3.3) we have

$$\begin{aligned}
\frac{\partial \widetilde{\mathrm{H}}}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t}\Big(\sqrt{2(T-t)}\,\mathrm{H}\Big) \\
&= \Big(2(T-t)\Big)^{3/2}(\Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2) - \sqrt{2(T-t)}\,\mathrm{H} \\
&= \widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 - \widetilde{\mathrm{H}}
\end{aligned}$$

and since $\widetilde{\nu} = \nu$,

$$\frac{\partial \widetilde{\nu}}{\partial s} = \left(\frac{ds}{dt}\right)^{-1} \frac{\partial \nu}{\partial t} = -2(T-t)\nabla \mathrm{H} = -\widetilde{\nabla}\widetilde{\mathrm{H}}\,.$$

Hence,

$$\begin{aligned}
\left|\frac{\partial}{\partial s}\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2\right| &= 2\left|\Big(\widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 - \widetilde{\mathrm{H}} + \langle \widetilde{\mathrm{H}}\widetilde{\nu} + y \,|\, \widetilde{\nu}\rangle - \langle y \,|\, \widetilde{\nabla}\widetilde{\mathrm{H}}\rangle\Big)\Big(\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\Big)\right| \\
&= 2\left|\widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 + \langle y \,|\, \widetilde{\nu}\rangle - \langle y \,|\, \widetilde{\nabla}\widetilde{\mathrm{H}}\rangle\right|\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right| \\
&\leq C(|y| + C)(|y| + C) \\
&\leq C(|y|^2 + 1)
\end{aligned}$$

for some constant $C$ independent of $s$.

Then,

$$\begin{aligned}
&\left|\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s\right| \qquad (3.2.9)\\
&\quad=\left|\int_M e^{-\frac{|y|^2}{2}}\left[\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\left(n-\widetilde{\mathrm{H}}^2-\langle y\,|\,\widetilde{\mathrm{H}}\widetilde{\nu}+y\rangle\right)+\frac{\partial}{\partial s}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\right]d\widetilde{\mu}_s\right|\\
&\quad\leq\int_M e^{-\frac{|y|^2}{2}}\left[C(|y|^2+1)(|y|^2+1)+C(|y|^2+1)\right]d\widetilde{\mu}_s\\
&\quad\leq C\int_M e^{-\frac{|y|^2}{2}}(|y|^4+1)\,d\widetilde{\mu}_s
\end{aligned}$$

and this last term is bounded uniformly in $s\in\left[-\frac{1}{2}\log T,+\infty\right)$ by a positive constant $C=C(\mathrm{Area}(\varphi_0),T)$ by the estimates in Stone's Lemma 3.2.7.
Supposing that there is a sequence of times $s_i\to+\infty$ such that

$$\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_{s_i}\geq\delta$$

for some $\delta>0$, then we have that in all the intervals $[s_i,s_i+\delta/(2C))$ such integral is larger than $\delta/2$. This is clearly in contradiction with the fact that

$$\int_{-\frac{1}{2}\log T}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s\,ds<+\infty\,,$$

stated in Proposition 3.2.5.
If $\widetilde{\varphi}_{s_i}$ is a locally smoothly converging subsequence of rescaled hypersurfaces (up to reparametrization), we have then that for every ball $B_R$,

$$\int_{\widetilde{\varphi}(M,s_i)\cap B_R} e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mathcal{H}}^n\leq\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_{s_i}\to 0\,,$$

hence, the limit hypersurface $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle=0$ at all its points.

Assume now that the initial hypersurface was embedded, then by Proposition 2.2.7 all the hypersurfaces $\widetilde{\varphi}_s$ are embedded and the only possibility for $\widetilde{M}_\infty$ not to be embedded is that two or more of its regions "touch" each other at some point $y\in\mathbb{R}^{n+1}$ with a common tangent space.
Let $g(t)$ be the metrics induced on the moving hypersurfaces; we consider the following set $\Omega_\varepsilon\subset M\times M\times[0,T)$ given by $\{(p,q,t)\,|\,d_t(p,q)\leq\varepsilon\sqrt{2(T-t)}\}$, where $d_t$ is the geodesic distance in the Riemannian manifold $(M,g(t))$. Let

$$B_\varepsilon=\inf_{\partial\Omega_\varepsilon}|\varphi(p,t)-\varphi(q,t)|/\sqrt{2(T-t)}\,,$$

we claim that $B_\varepsilon>0$ for any $\varepsilon>0$ small enough. Suppose that $B_\varepsilon=0$ for some $\varepsilon>0$; this means that there exists a sequence of times $t_i\nearrow T$ and points $p_i$, $q_i$

with $d_{t_i}(p_i, q_i) = \varepsilon\sqrt{2(T-t_i)}$ and $|\varphi(p_i, t_i) - \varphi(q_i, t_i)|/\sqrt{2(T-t_i)} \to 0$, hence, $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)| \to 0$ and $\widetilde{d}_i(p_i, q_i) = \varepsilon$, where we denoted by $\widetilde{\varphi}_i$ the rescaled hypersurfaces $\widetilde{\varphi}_i(p) = \frac{\varphi(p,t_i)-\varphi(p_i,t_i)}{\sqrt{2(T-t_i)}}$ and $\widetilde{d}_i = d_{t_i}/\sqrt{2(T-t_i)}$.

Reasoning as in the first part of this proof, by the uniform bound on the second fundamental form of the rescaled hypersurfaces, if $U^i$ is the connected component of $\widetilde{\varphi}_i^{-1}(B_{r_0}(\widetilde{\varphi}_i(p_i)))$ containing $p_i$, then $\widetilde{\varphi}_i(U^i)$ can be written as a graph of a smooth function $f_i$ over a subset of the tangent hyperplane to $\widetilde{\varphi}_i(M) \subset \mathbb{R}^{n+1}$ at the point $\widetilde{\varphi}_i(p_i)$. As $d_i(p_i, q_i) = \varepsilon$, when $\varepsilon > 0$ is small enough (depending on $r_0$ and $C_0$) the Lipschitz constants of these functions $f_i$ are uniformly bounded by a constant depending on $r_0$ and $C_0$; moreover, for every $i \in \mathbb{N}$ the point $q_i$ stays in $U^i$ and $\widetilde{\varphi}_i(q_i)$ belongs to the graph of $f_i$.

It is then easy to see that there exists a uniform positive bound from below on $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)|$, hence the constant $B_\varepsilon$ cannot be zero for such $\varepsilon > 0$.

Supposing that $\widetilde{M}_\infty$ has a self-intersection, we can parametrize it locally with a map $\widetilde{\varphi}_\infty : U \to \mathbb{R}^{n+1}$ such that a sequence of reparametrizations of the rescaled hypersurfaces $\widetilde{\varphi}_i$ converges to $\widetilde{\varphi}_\infty$ and $\widetilde{\varphi}_\infty(p) = \widetilde{\varphi}_\infty(q)$ for a couple of points $p, q \in U$.

Choosing $\varepsilon > 0$ smaller than the intrinsic distance between $p$ and $q$ in $\widetilde{M}_\infty$ and such that $B_\varepsilon > 0$, we consider the function

$$L(p, q, t) = |\varphi(p, t) - \varphi(q, t)|/\sqrt{2(T-t)}$$

on $\complement\Omega_\varepsilon \subset M \times M \times [0, T)$. If the minimum of $L$ at time $t$ is lower than $B_\varepsilon > 0$, then it cannot be attained on the boundary of $\Omega_\varepsilon$ and, arguing as in the proof of Proposition 2.2.7, such a minimum is nondecreasing. Hence, there is a positive lower bound on

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p, t) - \varphi(q, t)|/\sqrt{2(T-t)}\,.$$

Now we are done, since if we consider two sequences $p_i \to p$ and $q_i \to q$ we have definitely $\widetilde{d}_{t_i}(p_i, q_i) > \varepsilon$ and $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)| \to 0$, hence $d_{t_i}(p_i, q_i) > \varepsilon\sqrt{2(T-t_i)}$ which implies that $(p_i, q_i, t_i) \in \complement\Omega_\varepsilon$ and $|\varphi(p_i, t_i) - \varphi(q_i, t_i)|/\sqrt{2(T-t_i)} \to 0$, in contradiction with the previous conclusion. □

**Open Problem 3.2.11.** The limit hypersurface $\widetilde{M}_\infty$ is unique? That is, independent of the sequence $s_i \to +\infty$?

This problem is the parabolic analogue to the long-standing problem of uniqueness of the tangent cone in minimal surface theory.

We have seen in Proposition 1.4.1 that any of these limit hypersurfaces $\widetilde{M}_\infty$ satisfying $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$, that we call *homothetic*, generates a homothetically shrinking mean curvature flow given by $M_t = \widetilde{M}_\infty\sqrt{1-2t}$, vanishing at $T = 1/2$. As we said, few explicit examples are available: hyperplanes through the origin, the sphere $\mathbb{S}^n(\sqrt{n})$, the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and the Angenent's torus in [17].

**Open Problem 3.2.12.** Classify all the complete hypersurfaces (compact or not) satisfying $\mathrm{H} + \langle y \,|\, \nu \rangle = 0$, or at least the ones arising as blow-up limits of the flow of a compact and embedded hypersurface.

This problem is difficult; an equivalent formulation is to find the critical points of the Huisken's functional

$$\int_M e^{-\frac{|y|^2}{2}}\, d\mathcal{H}^n .$$

As we will see in the next sections, the classification is possible under the extra hypothesis $\mathrm{H} \geq 0$.

*Remark* 3.2.13. In the case of a homothetically shrinking hypersurface around a point $x_0 \in \mathbb{R}^{n+1}$ and vanishing at time $T$, the derivative in the monotonicity formula with the backward heat kernel $\rho_{x_0,T}$ is zero, that is, the integral

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t$$

is constant in time. Conversely, it follows from Proposition 1.4.1 and the uniqueness of the flow, that if such derivative is zero at some time, the hypersurface is homothetically shrinking around $x_0$ as at such time it must satisfy $\mathrm{H} + \frac{\langle x-x_0 \,|\, \nu\rangle}{2(T-t)} = 0$ for all its points.

Finally, notice that if the flow $\varphi_t$ is homothetically shrinking around $x_0$ for $t \in [0, T)$, the relative rescaled hypersurfaces $\widetilde{\varphi}_s = \frac{\varphi_t - x_0}{\sqrt{2(T-t)}}$ are not moving (as a subset of $\mathbb{R}^{n+1}$) and conversely.

*Remark* 3.2.14. By means of a slight modification of the function $\sigma$ defined in formula (3.2.3), it is possible to show that a compact *breather* for mean curvature flow is actually homothetically shrinking. A breather is a hypersurface moving by mean curvature such that $M_t = \lambda \mathrm{L}(M_{t'})$ for a couple of times $t > t'$, a constant $\lambda > 0$ and an isometry L of $\mathbb{R}^{n+1}$. Here and below we use the notation $M_s = \varphi(M, s)$.

Notice that *steady* or *expanding* compact breathers, that is when $\lambda = 1$ or $\lambda > 1$, do not exist by comparison with evolving spheres.

We can assume that $t' = 0$ and $t > 0$, for a compact hypersurface $M$ moving by mean curvature in $[0, T)$ and $\tau > 0$ we consider the function

$$\widetilde{\sigma}(M, \tau) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4\tau}}}{[4\pi\tau]^{n/2}}\, d\widetilde{\mathcal{H}}^n ,$$

then it is easy to see that $\widetilde{\sigma}(\mathrm{L}(M), \tau) = \widetilde{\sigma}(M, \tau)$ and for every $\lambda > 0$

$$\widetilde{\sigma}(\lambda M, \lambda^2 \tau) = \widetilde{\sigma}(M, \tau) . \tag{3.2.10}$$

Setting $\tau(t) = C - t$ for some $C > 0$ and integrating the analogue of equation (3.2.4) we have

$$\widetilde{\sigma}(M_0, \tau(0)) - \widetilde{\sigma}(M_t, \tau(t)) = \int_0^t \int_{M_s} \frac{e^{-\frac{|x - x_{\tau(s)}|^2}{4\tau(s)}}}{[4\pi\tau(s)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_{\tau(s)} \,|\, \nu \rangle}{2\tau(s)} \right|^2 d\widetilde{\mathcal{H}}^n \, ds \,.$$

By the rescaling property of $\widetilde{\sigma}$ in formula (3.2.10) and the hypothesis, we get

$$\widetilde{\sigma}(M_0, C) \geq \widetilde{\sigma}(M_t, C - t) = \widetilde{\sigma}(\lambda M_0, C - t) = \widetilde{\sigma}(M_0, (C - t)/\lambda^2) \,.$$

If now we choose $C = \frac{t}{1-\lambda^2} > t$ as $\lambda < 1$, we have $(C - t)/\lambda^2 = C$, and it follows that

$$\widetilde{\sigma}(M_0, \tau(0)) = \widetilde{\sigma}(M_0, C) = \widetilde{\sigma}(M_t, C - t) = \widetilde{\sigma}(M_t, \tau(t)) \,,$$

hence, by the formula above for almost every $s \in (0, t)$ we have $\mathrm{H} + \frac{\langle x - y_s \,|\, \nu \rangle}{2(C-s)}$ for some $y_s \in \mathbb{R}^{n+1}$. This clearly implies that we are dealing with a homothetically shrinking hypersurface.

We now fix a point $p \in M$ and consider a sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$, locally smoothly converging (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$ which satisfies $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu} \rangle = 0$ for every $y \in \widetilde{M}_\infty$.

We want to relate the limit heat density $\Theta(p)$ in Definition 3.2.3 with $\widetilde{M}_\infty$,

$$\begin{aligned}
\Theta(p) &= \lim_{t \to T} \theta(p, t) \\
&= \lim_{i \to \infty} \int_M \frac{e^{-\frac{|x - \widehat{p}|^2}{4(T - t(s_i))}}}{[4\pi(T - t(s_i))]^{n/2}} \, d\mu_{t(s_i)} \\
&= \lim_{i \to \infty} \int_M \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{n/2}} \, d\widetilde{\mu}_{s_i} \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}} \, d\widetilde{\mathcal{H}}^n \,,
\end{aligned}$$

where in the last passage we applied Corollary 3.2.8.

In particular, if $\widetilde{M}_\infty$ is a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$, then $\Theta(p) = \frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}} \, d\widetilde{\mathcal{H}}^n = 1$.

*Remark* 3.2.15. If we choose a time $\tau > 0$ which is strictly less than the maximal time $T$ of existence of the flow and we perform the rescaling procedure around the *nonsingular* point $\widehat{p} = \lim_{t \to \tau} \varphi(p, t) = \varphi(p, \tau)$, the hypersurface being regular around $p$ at time $\tau$, every limit of rescaled hypersurfaces must be flat, actually a union of hyperplanes through the origin. If moreover at $\varphi(p, \tau)$ the hypersurface has no self-intersections, such limit is a single hyperplane through the origin and

$$\lim_{t \to \tau} \int_M \frac{e^{-\frac{|x - \varphi(p, \tau)|^2}{4(\tau - t)}}}{[4\pi(\tau - t)]^{n/2}} \, d\mu_t = 1 \,.$$

This clearly holds for every $p \in M$ if the initial hypersurface is embedded.

*Remark* 3.2.16. By the previous remark, if $\tau \in (0,T)$ and $x_0 = \varphi_\tau(p)$ we have

$$\lim_{t\to\tau}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = 1$$

and

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4\tau}}}{[4\pi\tau]^{n/2}}\,d\mu_0 \geq 1$$

by the monotonicity formula, for every $p \in M$.
Then,

$$\mathrm{Area}(\varphi_0) \geq \int_M e^{-\frac{|x-x_0|^2}{4\tau}}\,d\mu_0 \geq [4\pi\tau]^{n/2}$$

and $\tau \leq [\mathrm{Area}]^{2/n}/(4\pi)$. As this holds for every $\tau < T$, we get the estimate $T \leq [\mathrm{Area}]^{2/n}/(4\pi)$ on the maximal time $T$ of existence of the flow (which is independent of the type I singularity hypothesis).

**Lemma 3.2.17 (White [123]).** *Among all the smooth, complete, hypersurfaces $M$ in $\mathbb{R}^{n+1}$ satisfying $\mathrm{H} + \langle y\,|\,\nu\rangle = 0$ and $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$, the hyperplanes with unit multiplicity through the origin are the only minimizers of the functional*

$$\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n\,.$$

*Hence, for all such hypersurfaces the value of this integral is at least* 1.

*Proof.* Suppose that there exists a smooth hypersurface $M = M_0$ such that

$$\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n \leq 1$$

and satisfies $\mathrm{H} + \langle y\,|\,\nu\rangle = 0$; then the flow $M_t = M\sqrt{1-2t}$ is a smooth mean curvature flow in the time interval $(-\infty, 1/2)$. Notice that for all the hypersurfaces $M_t$ the mean curvature H is bounded on compact subsets of $\mathbb{R}^{n+1}$, by the equation $\mathrm{H} + \langle y\,|\,\nu\rangle = 0$ and the definition of $M_t$.
Choosing a point $y_0 \in \mathbb{R}^{n+1}$ and a time $\tau \leq 1/2$ we consider the limit

$$\lim_{t\to-\infty}\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n\,,$$

where all the integrals are well defined since $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$.
Changing variables, we have

$$\lim_{t\to-\infty}\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n(y) = \lim_{t\to-\infty}\int_M \frac{e^{-\frac{|x\sqrt{1-2t}-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)/(1-2t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n(x)\,.$$

As $t \to -\infty$, the sequence of functions inside the integral pointwise converges to the function $e^{-\frac{|x|^2}{2}}/(2\pi)^{n/2}$ and they are definitely uniformly bounded from above, outside some large fixed ball $B_R(0) \subset \mathbb{R}^{n+1}$, by the function $e^{-|x|}$. Since this last function is integrable on $M$ by the hypothesis, using the dominated convergence theorem we get

$$\lim_{t\to-\infty} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\widetilde{\mathcal{H}}^n = \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|x|^2}{2}}\, d\widetilde{\mathcal{H}}^n \leq 1\,.$$

By the monotonicity formula this implies that

$$\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\widetilde{\mathcal{H}}^n \leq 1$$

for every $y_0 \in \mathbb{R}^{n+1}$ and $t < \tau \in (-\infty, 1/2)$.
Choosing now $y_0 \in M$ and $\tau = 0$, repeating the argument in Remark 3.2.15 (in this noncompact case it can be carried on by means of the hypothesis $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$) we have

$$\lim_{t\to0^-} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}}\, d\widetilde{\mathcal{H}}^n = 1\,,$$

hence, we conclude that the function

$$\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}}\, d\widetilde{\mathcal{H}}^n$$

is constant equal to 1 for every $t \in (-\infty, 0)$. Even if the evolving hypersurfaces $M_t$ are not compact, by the hypothesis $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$ it is straightforward to check (writing every integral as an integral on $M$ fixed) that the monotonicity formula still holds. Hence, we must have that the right-hand side of such formula is identically zero and $\mathrm{H}(y) + \frac{\langle y-y_0\,|\,\nu(y)\rangle}{-2t} = 0$ for every $t < 0$ and $y \in M_t$. Multiplying by $-2t$ and sending $t$ to zero, as H is bounded on compact subsets of $\mathbb{R}^{n+1}$ and $M_t \to M$, we conclude that $\langle y - y_0\,|\,\nu(y)\rangle = 0$ for every $y, y_0 \in M$. This condition easily implies that $M$ is a hyperplane through the origin of $\mathbb{R}^{n+1}$. $\square$

*Remark* 3.2.18. The smoothness hypothesis can be weakened in this lemma, provided that the set $M$ satisfies some definition of mean curvature to give sense to the condition $\mathrm{H} + \langle y\,|\,\nu\rangle = 0$ and coherent with the monotonicity formula (for instance, one can allow integral varifolds with bounded variation, see [21, 79]).
It is not known by the author whether the hypothesis $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$ can be removed. Anyway, it is satisfied by every limit hypersurface obtained as blow-up limit, by Corollary 3.2.8.

The following corollary is the consequence of Lemma 3.2.17 and the previous discussion about the relation between $\Theta$ and the limits of sequences of rescaled hypersurfaces.

**Corollary 3.2.19.** *The function $\Theta : M \to \mathbb{R}$ satisfies $\Theta \geq 1$ on all $M$. Moreover, if $\Theta(p) = 1$, every converging sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$ around $\widehat{p}$ converges to a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$.*
*It follows that $\Sigma \geq 1$.*

*Remark* 3.2.20. The fact that $\Theta \geq 1$ on all $M$ can also be proved directly using the argument in Remark 3.2.16. Since for every $\tau < T$ we have

$$\lim_{t\to\tau} \int_M \frac{e^{-\frac{|x-\varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t = 1$$

we get

$$\int_M \frac{e^{-\frac{|x-\varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t \geq 1$$

for every $t < \tau$. Keeping now $t < T$ fixed and sending $\tau \to T$ we have $\varphi_\tau(p) \to \widehat{p}$ and

$$\theta(p,t) = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t = \lim_{\tau\to T} \int_M \frac{e^{-\frac{|x-\varphi_\tau(p)via|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t \geq 1\,.$$

This clearly implies that $\Theta(p) = \lim_{t\to T} \theta(p,t) \geq 1$.

*Remark* 3.2.21. Rescaling around some $\widehat{p}$, by the discussion after Definition 3.2.2, means rescaling around some *reachable* point. Actually, we could rescale around *any* point $x_0 \in \mathbb{R}^{n+1}$ but if $x_0 \notin \mathcal{S}$, as the distance from $\varphi(M,t)$ and $x_0$ is definitely positive, the limit hypersurface is empty. This would imply that

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t \to 0$$

as $t \to T$.
By this corollary, if instead we consider $x_0 \in \mathcal{S}$, that is, $x_0 = \widehat{p}$ for some $p \in M$, there holds $\Theta(p) \geq 1$. Hence, there is a dichotomy between the points of $\mathbb{R}^{n+1}$, according to the value of the limit heat density function which can be either zero or at least one.
Moreover, by looking carefully at the first part of the proof of Lemma 3.2.17 we can see that this fact is independent of the type I hypothesis, it is indeed only a consequence of the upper semicontinuity of $\theta(p,t)$.

Actually, one can say more by the following result of White [123] (see also [35, Theorems 5.6, 5.7] and [115], moreover compare with [21, Theorem 6.11]), which also gives a partial answer to Problem 3.2.11.

**Theorem 3.2.22 (White [123]).** *There exist constants $\varepsilon = \varepsilon(n) > 0$ and $C = C(n)$ such that if $\Theta(p) < 1+\varepsilon$, then $|\mathrm{A}| \leq C(n)$ in a ball of $\mathbb{R}^{n+1}$ around $\widehat{p}$ uniformly in time $t \in [0,T)$.*

If the limit of a subsequence of rescaled hypersurfaces is a hyperplane through the origin, then $\Theta(p) = 1$ and by this theorem there is a ball around $\widehat{p}$ where the curvature is bounded. Then in such a ball, the *unscaled* hypersurfaces $\varphi_t$ (possibly after a reparametrization) converge locally uniformly in $C^0$ to some $\varphi_T$ with uniformly bounded curvature, this implies that the convergence is actually in $C^\infty$ by the interior estimates of Ecker and Huisken in [38]. Hence, it follows easily that the tangent hyperplane to $\varphi_T$ at the point $\widehat{p}$ coincides with the limit of *any* sequence of rescaled hypersurfaces, that is, there is full convergence and the limit hypersurface is unique, solving affirmatively Problem 3.2.11 in this very special case.

*Remark* 3.2.23. The strength of White's result is that it does not assume any condition on the sign of H and on the blow-up rate of the curvature. The theorem also holds without the type I hypothesis.

Another consequence is that there is a "gap" between the value 1 realized by the hyperplanes through the origin of $\mathbb{R}^{n+1}$ in the functional

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n$$

and any other smooth, complete hypersurface $M$ in $\mathbb{R}^{n+1}$ satisfying $\mathrm{H} + \langle y \,|\, \nu \rangle = 0$ and $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$.

## 3.3 Analysis of Singularities

**Definition 3.3.1.** We say that $p \in M$ is a *singular point* if there exists a sequence of points $p_i \to p$ in $M$ and times $t_i \to T$ such that for some constant $\delta > 0$ there holds

$$|\mathrm{A}(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T-t_i)}}\,.$$

We say that $p \in M$ is a *special singular point* if there exists a sequence of times $t_i \to T$ such that for some constant $\delta > 0$ there holds

$$|\mathrm{A}(p, t_i)| \geq \frac{\delta}{\sqrt{2(T-t_i)}}\,.$$

The lower estimate on the blow-up rate of the curvature

$$\max_{p\in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}\,.$$

and the compactness of $M$ clearly imply that there always exists at least one singular point, but not that any *special* singular point is present.

*Remark* 3.3.2. Though we will concentrate on the above definition of singular points at a type I singularity, it could possibly happen that while around some point $|\mathrm{A}|$ is blowing-up at least like $\frac{1}{\sqrt{2(T-t)}}$, at the same time in another region the hypersurface is getting singular with the curvature going to infinity at a lower rate. This "bad" behavior is actually excluded, at least in the case of a flow of hypersurfaces with nonnegative mean curvature, by the analysis of Stone [115, Section 4].

Suppose that $p \in M$ is a special singular point, then after rescaling the hypersurface as before around $\widehat{p}$ we have for $s_i = -\frac{1}{2}\log{(T-t_i)}$,

$$|\widetilde{\mathrm{A}}(p,s_i)| = \sqrt{2(T-t_i)}|\mathrm{A}(p,t_i)| \geq \delta > 0$$

which implies that taking a subsequence of $s_i \to +\infty$, any limit hypersurface obtained by Proposition 3.2.10 cannot be flat as $\widetilde{\mathrm{A}} \neq 0$ at some point in the ball $B_{C_0\sqrt{2n}}$.

If $p \in M$ is not a special singular point clearly, for every sequence $s_i \to +\infty$,

$$|\widetilde{\mathrm{A}}(p,s_i)| = \sqrt{2(T-t_i)}|\mathrm{A}(p,t_i)| \to 0,$$

that is, any limit hypersurface satisfies $\widetilde{\mathrm{A}} = 0$ at some point in the ball $B_{C_0\sqrt{2n}}$.

**Open Problem 3.3.3.** Is it true that any limit hypersurface associated to a non-special singular point is a union of hyperplanes through the origin?
This conclusion would follow if any nonflat hypersurface $M$ satisfying $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$ and $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$ cannot have a point where the second fundamental form is zero.

By means of a small variation of an argument by Stone, we have a good description when the limit hypersurface is a single hyperplane.

**Proposition 3.3.4 (Stone [115]).** *If the limit of rescaled hypersurfaces around $\widehat{p}$ is a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$, or equivalently by* Lemma 3.2.17 *there holds $\Theta(p) = 1$, then $p$ cannot be a singular point.*

*Proof.* By Corollary 3.2.19, the point $p \in M$ is a minimum of $\Theta : M \to \mathbb{R}$ which is an upper semicontinuous function. Hence $p$ is actually a continuity point for $\Theta$. We want to show that for every sequence $p_i \to p$ and $t_i \to T$ we have $\theta(p_i, t_i) \to 1 = \Theta(p)$.
Suppose that there exists $\delta > 0$ such that $\theta(p_i, t_i) \to 1 + \delta$. For every $j \in \mathbb{N}$ there exists $i_0$ such that $t_i \geq t_j$ for every $i > i_0$, hence $\theta(p_i, t_i) \leq \theta(p_i, t_j)$. Sending $i \to \infty$ we then get $1 + \delta \leq \theta(p, t_j)$. This is clearly a contradiction, as sending now $j \to \infty$, we have $\theta(p, t_j) \to \Theta(p) = 1$ (what we did is closely related to Dini's theorem on monotone convergence of continuous functions).
If $p$ is a singular point with $p_i \to p$ and $t_i \to T$ such that for some constant $\delta > 0$ there holds $|\mathrm{A}(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T-t_i)}}$, we consider the families of rescaled

hypersurfaces around $\widehat{p}_i$,

$$\widetilde{\varphi}_i(q,s) = \frac{\varphi(q,t) - \widehat{p}_i}{\sqrt{2(T-t)}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

with associated measures $\widetilde{\mu}_{i,s}$ and we set

$$\psi_i(q) = \widetilde{\varphi}_i(q,s_i) = \frac{\varphi(q,t_i) - \widehat{p}_i}{\sqrt{2(T-t_i)}} \qquad s_i = -\frac{1}{2}\log(T-t_i)\,,$$

with associated measures $\widetilde{\mu}_{i,s_i}$.
For every $\varepsilon > 0$, as $\Theta(p_i) \geq 1$ we have definitely

$$\begin{aligned}
\varepsilon \geq \theta(p_i,t_i) - 1 \geq \theta(p_i,t_i) - \Theta(p_i) \\
&= \int_M \frac{e^{-\frac{|x-\widehat{p}_i|^2}{4(T-t_i)}}}{[4\pi(T-t_i)]^{n/2}}\,d\mu_{t_i} - \Theta(p_i) \\
&= \frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{i,s_i} - \Theta(p_i) \\
&\geq \int_{s_i}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_{i,s}\,ds\,.
\end{aligned}$$

Hence, since by the uniform curvature estimates of Proposition 3.2.9, see computation (3.2.9), we have,

$$\left|\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_s\right| \leq C$$

where $C = C(\mathrm{Area}(\varphi_0), T)$ is a positive constant independent of $s$, we get

$$\begin{aligned}
\varepsilon &\geq \int_{s_i}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_{i,s}\,ds \\
&\geq \frac{1}{2}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mu}_{i,s_i}
\end{aligned}$$

possibly passing to a subsequence of $s_i$ (not relabelled).
If now we proceed as in Proposition 3.2.10 and we extract from the sequence of hypersurfaces $\psi_i$ a locally smoothly converging subsequence (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$, by Lemma 3.2.7 we have

$$\varepsilon \geq \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,d\widetilde{\mathcal{H}}^n$$

for every $\varepsilon > 0$, hence $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$.

Finally, by Corollary 3.2.8,

$$\frac{1}{(2\pi)^{n/2}}\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n = \lim_{i\to\infty}\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{i,s_i} = \lim_{i\to\infty}\theta(p_i,t_i) = 1$$

then, by Lemma 3.2.17, the hypersurface $\widetilde{M}_\infty$ has to be a hyperplane. But since all the points $\psi_i(p_i)$ belong to the ball of radius $C_0\sqrt{2n}\subset\mathbb{R}^{n+1}$ and the second fundamental form $\mathrm{A}^i$ of $\psi_i$ satisfies $|\mathrm{A}^i(p_i)|\geq\delta>0$ for every $i\in\mathbb{N}$, by the hypothesis, it follows that the second fundamental form of $\widetilde{M}_\infty$ is not zero at some point in the ball $B_{C_0\sqrt{2n}}(0)$.
Since we have a contradiction, $p$ cannot be a singular point of the flow. □

*Remark* 3.3.5. This lemma is an immediate consequence of White's Theorem 3.2.22, but we wanted to emphasize the fact that the only really needed "ingredient" by the line of analysis of Stone is the uniqueness of the hyperplanes as minimizers of the integral $\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n$ among the hypersurfaces satisfying $\mathrm{H}+\langle y\,|\,\nu\rangle=0$ (Lemma 3.2.17).

**Open Problem 3.3.6.** To the author's knowledge, even if we are dealing with the flow of embedded hypersurfaces, the existence of at least one special singular point is an open problem.
A related stronger statement would be that *every singular point is a special singular point.*

This problem and Problem 3.3.3, in the embedded situation are quite connected. Indeed, if the initial hypersurface is embedded any limit hypersurface $\widetilde{M}_\infty$ is also embedded, so the union of two or more hyperplanes cannot arise.
This means that if Problem 3.3.3 has a positive answer, for every nonspecial singular point $p\in M$ the limit hypersurfaces can be only single unit multiplicity hyperplane through the origin, hence, by Proposition 3.3.4 the point is actually not singular.
Then, if there are no special singular points, it follows that there are no singular points at all which is a contradiction and also Problem 3.3.6 has a positive answer.
Repeating this argument in the general nonembedded case, unfortunately one could obtain a *union* of hyperplanes, or even more disturbing, hyperplanes with integer multiplicities higher than one.

Putting together Propositions 3.2.10 and 3.3.4 we have the following conclusion, as the only flat blow-up limits of a mean curvature flow of an embedded hypersurface are the hyperplanes through the origin of $\mathbb{R}^{n+1}$.

**Corollary 3.3.7.** *At a singular point $p\in M$ a limit $\widetilde{M}_\infty$ of rescaled hypersurfaces is a smooth, nonempty, complete hypersurface with bounded local volume and bounded curvature with all its covariant derivatives, which satisfies $\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle=0$ and it is not a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$.*
*Moreover, if the initial hypersurface is embedded, $\widetilde{M}_\infty$ is also embedded and nonflat.*

*Remark* 3.3.8. Another line to obtain a nontrivial homothetic blow-up limit is to apply, instead of Stone's argument, White's Theorem 3.2.22 excluding the presence of singularities in the case $\Sigma = 1$ (recall Definition 3.2.3).

Indeed, as the set of reachable points $\mathcal{S}$ is compact, if $\Sigma > 1$ there must exist a point $x_0 = \widehat{p}$ such that $\Theta(p) > 1$, otherwise, by a covering argument and White's theorem we can conclude that the curvature is uniformly bounded as $t \to T$, which is a contradiction. Rescaling the hypersurfaces around $x_0 \in \mathbb{R}^{n+1}$ we then get a homothetic blow-up limit which cannot be a single unit multiplicity hyperplane through the origin.

*Remark* 3.3.9. Finally, we can produce a homothetic blow-up limit also by rescaling the hypersurfaces around some *moving* points as follows. For every $t \in [0, T)$, we let $x_t$ be any maximum point realizing $\sigma(t)$ in Definition 3.2.3, that is

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t$$

and we consider the rescaled hypersurfaces with associated measures $\widetilde{\mu}_s$ around the points $x_t$,

$$\widetilde{\varphi}(q, s) = \frac{\varphi(q, t(s)) - x_{t(s)}}{\sqrt{2(T - t(s))}}, \qquad s = s(t) = -\frac{1}{2}\log(T-t)\,.$$

Rescaling also the *maximal* monotonicity formula (3.2.4) we have

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \leq 0\,.$$

It follows that, integrating this formula as before, we get

$$\sigma(0) - \Sigma = \int_{-\frac{1}{2}\log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\, ds < +\infty\,,$$

and with the same argument we can produce a homothetic limit hypersurface $\widetilde{M}_\infty$ such that

$$\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n = \Sigma \geq 1\,.$$

Since when $\Sigma = 1$ by White's theorem the curvature is bounded, the limit hypersurface $\widetilde{M}_\infty$ cannot be a single unit multiplicity hyperplane through the origin.

## 3.4 Hypersurfaces with Nonnegative Mean Curvature

For most of this section we assume that all the hypersurfaces are embedded. At the end we will discuss the general immersed case.

If the compact initial hypersurface is embedded and has $\mathrm{H} \geq 0$ (or at some positive time the evolving hypersurface becomes like that) then the analysis of the previous section can be pushed forward since we can restrict the class of the possible limits of rescaled hypersurfaces to the ones satisfying these two conditions. In this case *every* singular point is a special singular point and it is actually possible to classify all the embedded limit hypersurfaces in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ and $\mathrm{H} \geq 0$, see [67, 68] and [116].

It will follow that Problem 3.3.3 and consequently Problem 3.3.6 have a positive answer.

Finally, as we mentioned in Remark 3.3.2, it cannot happen that the curvature in some region of the hypersurface goes to infinity with a rate lower than $\frac{1}{\sqrt{2(T-t)}}$.

We recall that if $\mathrm{H} \geq 0$ for the initial hypersurface, after a positive time $t_0 > 0$, there exists a constant $\alpha > 0$ such that $\alpha|\mathrm{A}| \leq \mathrm{H} \leq n|\mathrm{A}|$ everywhere on $M$ for every time $t \geq t_0$, by Corollary 2.4.3.

Hence, we can assume in the sequel that for $t \in [0, T)$ we have

$$\frac{\alpha}{\sqrt{2(T-t)}} \leq \max_{p \in M} \mathrm{H}(p,t) \leq \frac{C}{\sqrt{2(T-t)}} \,.$$

**Proposition 3.4.1 (Huisken [67, 68], Abresch and Langer [1] in the one-dimensional case).** *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, complete, embedded, mean convex hypersurface in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ at every $x \in M$ and there exists a constant $C$ such that $|\mathrm{A}| + |\nabla \mathrm{A}| \leq C$ and $\mathcal{H}^n(M \cap B_R) \leq Ce^R$, for every ball of radius $R > 0$ in $\mathbb{R}^{n+1}$.*

*Then, up to a rotation of $\mathbb{R}^{n+1}$, $M$ must be one of only $n+1$ possible hypersurfaces, namely, either a hyperplane through the origin or the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.*

*In the special one-dimensional case the only smooth, complete, embedded curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle x \,|\, \nu \rangle = 0$ are the lines through the origin and the unit circle (notice that in this case neither the positivity of the curvature nor the two estimates above are assumed).*

*Proof.* Let us suppose that $M$ is connected. If the theorem is true in this case, it it easy to see that it is not possible to have a nonconnected embedded hypersurface satisfying the hypotheses. Indeed, any connected component has to belong to the list of the statement and every two hypersurfaces in such list either coincide or have some intersections.

We first deal separately with the case $n = 1$.

Fixing a reference point on a curve $\gamma$ we have an arclength parameter $s$ which gives a unit tangent vector field $\tau = \gamma_s$ and a unit normal vector field $\nu$, which is the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector $\tau$; then the curvature is given by $k = \langle \partial_s \tau \,|\, \nu \rangle = -\langle \tau \,|\, \partial_s \nu \rangle$.
The relation $k = -\langle \gamma \,|\, \nu \rangle$ implies the ODE for the curvature $k_s = k \langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then also $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODE's we conclude that $k$ is identically zero and we are dealing with a line $L$ which, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in L$, must contain the origin of $\mathbb{R}^2$.
So we suppose that $k$ is always nonzero and possibly reversing the orientation of the curve, we assume that $k > 0$ at every point, that is, the curve is strictly convex.
Computing the derivative of $|\gamma|^2$,

$$\partial_s |\gamma|^2 = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\partial_s \log k$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C > 0$, it follows that $k$ is bounded from below by $C > 0$.
We consider now a new coordinate $\theta = \arccos \langle e_1 \,|\, \nu \rangle$, this can be done for the whole curve as we know that this latter is convex (obviously, as for the arclength parameter $s$ it is only locally continuous, $\theta$ "jumps" after a complete round).
Differentiating with respect to the arclength parameter we have $\partial_s \theta = k$ and

$$k_\theta = k_s/k = \langle \gamma \,|\, \tau \rangle \qquad k_{\theta\theta} = \frac{\partial_s k_\theta}{k} = \frac{1 + k\langle \gamma \,|\, \nu \rangle}{k} = \frac{1}{k} - k\,. \tag{3.4.1}$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\partial_\theta[k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant $E$ along all the curve. Notice that such a quantity $E$ cannot be less than 1; moreover, if $E = 1$ we have that $k$ must be constant and equal to 1 along the curve, which consequently must be the unit circle centered at the origin of $\mathbb{R}^2$.
When $E > 1$, it follows that $k$ is uniformly bounded from above, hence recalling that $k = Ce^{|\gamma|^2/2}$, the image of the curve is contained in a ball of $\mathbb{R}^2$ and by the embeddedness and completeness hypotheses, the curve must be closed, simple and strictly convex, as $k > 0$ at every point.
We now suppose that $\gamma$ is not the unit circle and we look at the critical points of the curvature $k$. Since $k_{\theta\theta} = \frac{1}{k} - k$, it holds that $k_{\theta\theta} \neq 0$ when $k_\theta = 0$, otherwise this second-order ODE for $k$ would imply $k_\theta = 0$ everywhere, hence $k = 1$ identically and we would be in the case of the unit circle. Thus, the critical points of the curvature are not degenerate, hence, by the compactness of the curve they are isolated and finite. Moreover, by looking at the equation for the curvature (3.4.1) we can see easily that $k_{\min} < 1$ and $k_{\max} > 1$.
Suppose now that $k(0) = k_{\max}$ and $k(\overline{\theta})$ is the first subsequent critical value for $k$, for some $\overline{\theta} > 0$. Then the curvature is strictly decreasing in the interval $[0, \overline{\theta}]$

and again by the second-order ODE, the function $k$ (hence also the curve, by integration) is symmetric with respect to $\theta = 0$ and $\theta = \overline{\theta}$. This clearly implies that $k(\overline{\theta})$ must be the minimum $k_{\min}$ of the curvature, as every critical point is not degenerate.

By the four-vertex theorem [92, 100], on every closed curve there are at least four critical points of $k$, as a consequence our curve is composed of at least four pieces like the one described above. Hence, since the curve is closed and embedded the curvature $k(\theta)$ must be a periodic function with period $T > 0$ not larger than $\pi$ (since $2\pi$ is an obvious multiple of the period) and $\overline{\theta} = T/2$. More precisely, the period $T$ must be $2\pi/n$ for some $n \geq 2$.

By a straightforward computation, starting by differentiating the equation $k_{\theta\theta} = \frac{1}{k} - k$, one gets $(k^2)_{\theta\theta\theta} + 4(k^2)_\theta = 4k_\theta/k$; then we compute

$$\begin{aligned}
4\int_0^{T/2} \sin 2\theta \,\frac{k_\theta}{k}\, d\theta &= \int_0^{T/2} \sin 2\theta \,[(k^2)_{\theta\theta\theta} + 4(k^2)_\theta]\, d\theta \\
&= \sin 2\theta (k^2)_{\theta\theta}\big|_0^{T/2} - 2\int_0^{T/2} \cos 2\theta (k^2)_{\theta\theta}\, d\theta + 4\int_0^{T/2} \sin 2\theta (k^2)_\theta \, d\theta \\
&= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] - 2\cos 2\theta (k^2)_\theta\big|_0^{T/2} \\
&\quad - 4\int_0^{T/2} \sin 2\theta (k^2)_\theta\, d\theta + 4\int_0^{T/2} \sin 2\theta (k^2)_\theta \, d\theta \\
&= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] - 4\cos T k(T/2)k_\theta(T/2) + 4k(0)k_\theta(0)\,.
\end{aligned}$$

Now, since $k_\theta(0) = k_\theta(T/2) = 0$ and $k(T/2) = k_{\min}$, using the equation for the curvature $k_{\theta\theta} = 1/k - k$ we get

$$4\int_0^{T/2} \sin 2\theta \,\frac{k_\theta}{k}\, d\theta = 2\sin T(1 - k_{\min}^2)\,,$$

and this last term is nonnegative as $k_{\min} < 1$ and $0 < T \leq \pi$.
Looking at the left-hand integral we see instead that the factor $\sin 2\theta$ is always nonnegative, since $T \leq \pi$ and $k_\theta$ is always nonpositive in the interval $[0, T/2]$, as we assumed that we were moving from the maximum $k_{\max}$ at $\theta = 0$ to the minimum $k_{\min}$ at $\theta = T/2$ without crossing any other critical point of $k$. This gives a contradiction so $\gamma$ must be the unit circle.

We suppose now that $n \geq 2$.
By covariant differentiation of the equation $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$ in an orthonormal frame $\{e_1, \dots, e_n\}$ on $M$ we get

$$\nabla_i \mathrm{H} = \langle x \,|\, e_k\rangle h_{ik},$$

$$\nabla_i\nabla_j \mathrm{H} = h_{ij} + \langle x \,|\, \nu\rangle h_{ik}h_{jk} + \langle x \,|\, e_k\rangle \nabla_i h_{jk} = h_{ij} - \mathrm{H} h_{ik}h_{jk} + \langle x \,|\, e_k\rangle \nabla_k h_{ij} \quad (3.4.2)$$

where we used Gauss–Weingarten and Codazzi equations (1.1.1), (1.1.3).

Contracting now with $g^{ij}$ and $h^{ij}$ respectively we have

$$\Delta \mathrm{H} = \mathrm{H} - \mathrm{H}|\mathrm{A}|^2 + \langle x \,|\, e_k \rangle \nabla_k \mathrm{H} = \mathrm{H}(1 - |\mathrm{A}|^2) + \langle x \,|\, \nabla \mathrm{H} \rangle, \tag{3.4.3}$$

$$h^{ij} \nabla_i \nabla_j \mathrm{H} = |\mathrm{A}|^2 - \mathrm{tr}(\mathrm{A}^3)\mathrm{H} + \langle x \,|\, e_k \rangle \nabla_k |\mathrm{A}|^2/2,$$

which implies, by Simons' identity (1.1.4),

$$\Delta |\mathrm{A}|^2 = 2|\mathrm{A}|^2(1 - |\mathrm{A}|^2) + 2|\nabla \mathrm{A}|^2 + \langle x \,|\, \nabla |\mathrm{A}|^2 \rangle\,.$$

From equation (3.4.3) and the strong maximum principle for elliptic equations we see that, since $M$ satisfies $\mathrm{H} \geq 0$ by assumption and $\Delta \mathrm{H} \leq \mathrm{H} + \langle x \,|\, \nabla \mathrm{H} \rangle$, we must either have that $\mathrm{H} \equiv 0$ or $\mathrm{H} > 0$ on all $M$.

The case $\mathrm{H} \equiv 0$ can be easily handled: as $M$ is complete and $x$ is a tangent vector field on $M$ by the equation $\langle x \,|\, \nu \rangle = 0$, for every point $x_0$ of $M$ there is a unique solution of the ODE $\gamma'(s) = x(\gamma(s)) = \gamma(s)$ passing through $x_0$ and contained in $M$ for every $s \in \mathbb{R}$, but such solution is simply the line in $\mathbb{R}^{n+1}$ passing through $x_0$ and the origin. Thus, $M$ has to be a cone and being smooth the only possibility is a hyperplane through the origin of $\mathbb{R}^{n+1}$.

Therefore we may assume that the mean curvature satisfies the strict inequality $\mathrm{H} > 0$ everywhere (so dividing by $\mathrm{H}$ and $|\mathrm{A}|$ is allowed).

Now let $R > 0$ and define $\eta$ to be the inward unit conormal to $M \cap B_R(0)$ along $\partial\big(M \cap B_R(0)\big)$, which is a smooth boundary for almost every $R > 0$ (by Sard's theorem). Then, supposing that $R$ belongs to the set $\Omega \subset \mathbb{R}^+$ of the *regular values* of the function $|\cdot|$ restricted to $M \subset \mathbb{R}^{n+1}$, from equation (3.4.3) and the divergence theorem we compute

$$\begin{aligned}
\varepsilon_R &= -\int_{\partial(M \cap B_R(0))} |\mathrm{A}| \langle \nabla \mathrm{H} \,|\, \eta \rangle e^{-R^2/2}\, d\mathcal{H}^{n-1}(x) \\
&= \int_{M \cap B_R(0)} |\mathrm{A}| \Delta \mathrm{H} e^{-|x|^2/2} + \langle \nabla (|\mathrm{A}| e^{-|x|^2/2}) \,|\, \nabla \mathrm{H} \rangle \, d\mathcal{H}^n(x) \\
&= \int_{M \cap B_R(0)} \Big( |\mathrm{A}|\mathrm{H}(1 - |\mathrm{A}|^2) + |\mathrm{A}| \langle x \,|\, \nabla \mathrm{H} \rangle \Big) e^{-|x|^2/2}\, d\mathcal{H}^n(x) \\
&\quad + \int_{M \cap B_R(0)} \Big( \frac{1}{2|\mathrm{A}|} \langle \nabla |\mathrm{A}|^2 \,|\, \nabla \mathrm{H} \rangle - |\mathrm{A}| \langle x \,|\, \nabla \mathrm{H} \rangle \Big) e^{-|x|^2/2}\, d\mathcal{H}^n(x) \\
&= \int_{M \cap B_R(0)} \Big( |\mathrm{A}|\mathrm{H}(1 - |\mathrm{A}|^2) + \frac{1}{2|\mathrm{A}|} \langle \nabla |\mathrm{A}|^2 \,|\, \nabla \mathrm{H} \rangle \Big) e^{-|x|^2/2}\, d\mathcal{H}^n(x)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\delta_R &= -\int_{\partial(M \cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|} \langle \nabla |\mathrm{A}|^2 \,|\, \eta \rangle e^{-R^2/2}\, d\mathcal{H}^{n-1}(x) \\
&= \int_{M \cap B_R(0)} \frac{\mathrm{H}}{|\mathrm{A}|} \Delta |\mathrm{A}|^2 e^{-|x|^2/2} + \Big\langle \nabla \Big( \frac{\mathrm{H}}{|\mathrm{A}|} | e^{-|x|^2/2} \Big) \,\Big|\, \nabla |\mathrm{A}|^2 \Big\rangle \, d\mathcal{H}^n(x)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{M\cap B_R(0)} \Big(2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{2\mathrm{H}|\nabla \mathrm{A}|^2}{|\mathrm{A}|} + \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle\Big) e^{-|x|^2/2}\, d\mathcal{H}^n(x)\\
&\quad + \int_{M\cap B_R(0)} \Big(\frac{\langle \nabla \mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3} - \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle\Big) e^{-|x|^2/2}\, d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{2\mathrm{H}|\nabla \mathrm{A}|^2}{|\mathrm{A}|} + \frac{\langle \nabla \mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3}\Big)\\
&\qquad \times e^{-|x|^2/2}\, d\mathcal{H}^n(x)\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma_R = 2\delta_R - 4\varepsilon_R &= \int_{M\cap B_R(0)} \left(\frac{4\mathrm{H}|\nabla \mathrm{A}|^2}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{|\mathrm{A}|^3}\right) e^{-|x|^2/2}\, d\mathcal{H}^n(x)\\
&= \int_{M\cap B_R(0)} \Big(4|\mathrm{A}|^2|\nabla \mathrm{A}|^2 - |\nabla|\mathrm{A}|^2|^2\Big) \frac{\mathrm{H}}{|\mathrm{A}|^3} e^{-|x|^2/2}\, d\mathcal{H}^n(x)\,.
\end{aligned}
$$

As we have $4|\mathrm{A}|^2|\nabla \mathrm{A}|^2 \geq |\nabla|\mathrm{A}|^2|^2$, this quantity $\sigma_R$ is nonnegative and nondecreasing in $R$.

If now we show that $\liminf_{R\to+\infty} \sigma_R = 0$ we can conclude that, at every point of $M$,

$$
4|\mathrm{A}|^2|\nabla \mathrm{A}|^2 = |\nabla|\mathrm{A}|^2|^2\,. \tag{3.4.4}
$$

Getting back to the definitions of $\varepsilon_R$ and $\delta_R$, we have

$$
\begin{aligned}
|\sigma_R| &= \Bigg| -2\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle \nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\, d\mathcal{H}^{n-1}\\
&\qquad + 4\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle \nabla \mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\, d\mathcal{H}^{n-1}\Bigg|\\
&\leq 4e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}|\nabla|\mathrm{A}|^2| + |\mathrm{A}||\nabla \mathrm{H}|\, d\mathcal{H}^{n-1}\\
&\leq 8e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \mathrm{H}|\nabla \mathrm{A}| + |\mathrm{A}||\nabla \mathrm{H}|\, d\mathcal{H}^{n-1}\\
&\leq Ce^{-R^2/2}\mathcal{H}^{n-1}(\partial(M\cap B_R(0)))\,,
\end{aligned}
$$

by the estimates on $\mathrm{A}$ and $\nabla \mathrm{A}$ in the hypotheses.

Now, suppose that for every $R$ belonging to the set $\Omega \subset \mathbb{R}^+$ (which is of full measure) and $R$ larger than some $R_0 > 0$ we have

$$
\mathcal{H}^{n-1}(\partial(M\cap B_R(0))) \geq \delta R e^{R^2/4}
$$

for some constant $\delta > 0$. Setting $x^M$ to be the projection of the vector $x$ on the tangent space to $M$, as the function $R \mapsto \mathcal{H}^n(M\cap B_R(0))$ is monotone and continuous from the left and actually continuous at every value $R \in \Omega$, we can differentiate it almost everywhere in $\mathbb{R}^+$ and we have (by the coarea formula, see [41] or [105]),

$$\begin{aligned}\mathcal{H}^n(M \cap B_R(0)) - \mathcal{H}^n(M \cap B_r(0)) &\geq \int_r^R \frac{d}{d\xi}\mathcal{H}^n(M \cap B_\xi(0))\, d\xi \\ &\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} |\nabla^M |x||^{-1}\, d\mathcal{H}^{n-1}(x)\, d\xi \\ &= \int_r^R \int_{\partial(M\cap B_\xi(0))} |x|/|x^M|\, d\mathcal{H}^{n-1}(x)\, d\xi \\ &\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mathcal{H}^{n-1}(x)\, d\xi\,,\end{aligned}$$

where the derivative in the integral is taken only at the points where it exists and $\nabla^M|x|$ denotes the projection of the $\mathbb{R}^{n+1}$-gradient of the function $|x|$ on the tangent space to $M$.

Hence, if $R > r > R_0$ we get

$$\begin{aligned}\mathcal{H}^n(M \cap B_R(0)) - \mathcal{H}^n(M \cap B_r(0)) &\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mathcal{H}^{n-1}\, d\xi \\ &\geq \delta \int_r^R \xi e^{\xi^2/4}\, d\xi \\ &= 2\delta(e^{R^2/4} - e^{r^2/4})\,,\end{aligned}$$

then if $R$ goes to $+\infty$, the quantity $\mathcal{H}^n(M \cap B_R(0))e^{-R}$ diverges, in contradiction with the hypotheses of the proposition. Hence, the $\liminf$ of the quantity $e^{-R^2/4}\mathcal{H}^{n-1}(\partial(M \cap B_R(0)))$ as $R \to +\infty$ in the set $\Omega$ has to be zero. It follows that the same holds for $|\sigma_R|$ and equation (3.4.4) is proved.

Making explicit such equation, by the equality condition in the Cauchy–Schwartz inequality it immediately follows that at every point there exist constants $c_k$ such that

$$\nabla_k h_{ij} = c_k h_{ij}$$

for every $i, j$. Contracting this equation with the metric $g^{ij}$ and with $h^{ij}$ we get $\nabla_k \mathrm{H} = c_k \mathrm{H}$ and $\nabla_k |\mathrm{A}|^2 = 2c_k |\mathrm{A}|^2$, hence $\nabla_k \log \mathrm{H} = c_k$ and $\nabla_k \log |\mathrm{A}|^2 = 2c_k$.

This implies that locally $|\mathrm{A}| = \alpha \mathrm{H}$ for some constant $\alpha > 0$ and by connectedness this relation has to hold globally on $M$.

Suppose now that at a point $|\nabla \mathrm{H}| \neq 0$, then $\nabla_k h_{ij} = c_k h_{ij} = \frac{\nabla_k \mathrm{H}}{\mathrm{H}} h_{ij}$ which is a symmetric 3-tensor by the Codazzi equations (1.1.3), hence $\nabla_k \mathrm{H} h_{ij} = \nabla_j \mathrm{H} h_{ik}$. Computing then in normal coordinates with an orthonormal basis $\{e_1, \ldots, e_n\}$

such that $e_1 = \nabla \mathrm{H}/|\nabla \mathrm{H}|$, we have

$$0 = |\nabla_k \mathrm{H} h_{ij} - \nabla_j \mathrm{H} h_{ik}|^2 = 2|\nabla \mathrm{H}|^2 \left( |\mathrm{A}|^2 - \sum_{i=1}^{n} h_{1i}^2 \right) .$$

Hence, $|\mathrm{A}|^2 = \sum_{i=1}^{n} h_{1i}^2$ and

$$|\mathrm{A}|^2 = h_{11}^2 + 2\sum_{i=2}^{n} h_{1i}^2 + \sum_{i,j\neq 1}^{n} h_{ij}^2 = h_{11}^2 + \sum_{i=2}^{n} h_{1i}^2$$

so $h_{ij} = 0$ unless $i = j = 1$, which means that A has rank one.
Thus, we have two possible (not mutually excluding) situations at every point of $M$, either A has rank one or $\nabla \mathrm{H} = 0$.

If the kernel of A is empty everywhere, A must have rank at least two as we assumed $n \geq 2$, then we have $\nabla \mathrm{H} = 0$ which implies $\nabla \mathrm{A} = 0$ and $h_{ij} = \mathrm{H} h_{ik} h_{kj}$, by equation (3.4.2). This means that all the eigenvalues of A are 0 or $1/\mathrm{H}$. As the kernel is empty $\mathrm{A} = \mathrm{H} g/n$, precisely $\mathrm{H} = \sqrt{n}$ and $\mathrm{A} = g/\sqrt{n}$. Then, the complete hypersurface $M$ has to be the sphere $\mathbb{S}^n(\sqrt{n})$, indeed we compute

$$\Delta^M |x|^2 = 2n + 2\langle x \,|\, \Delta^M x\rangle = 2n + 2\mathrm{H}\langle x \,|\, \nu\rangle = 2n - 2\mathrm{H}^2 = 0 \,,$$

by means of the structural equation $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$, hence $|x|^2$ is a harmonic function on $M$ and looking at the point of $M$ of minimum distance from the origin, by the strong maximum principle for elliptic equations, it must be constant on $M$.

We assume now that the kernel of A is not empty at some point $p \in M$ and let $v_1(p), \dots, v_{n-m}(p) \in T_pM \subset \mathbb{R}^{n+1}$ be a family of unit orthonormal tangent vectors spanning such $(n-m)$-dimensional kernel, that is $h_{ij}(p) v_k^j(p) = 0$. Then, the geodesic $\gamma(s)$ from $p$ in $M$ (which is complete) with initial velocity $v_k(p)$ satisfies

$$\nabla_s (h_{ij} \gamma_s^j) = \mathrm{H}^{-1} \langle \nabla \mathrm{H} \,|\, \gamma_s\rangle h_{ij} \gamma_s^j$$

hence, by Gronwall's lemma there holds $h_{ij}(\gamma(s))\gamma_s^j(s) = 0$ for every $s \in \mathbb{R}$.
Since $\gamma$ is a geodesic in $M$, the normal to the curve in $\mathbb{R}^{n+1}$ is also the normal to $M$, then letting $\kappa$ be the curvature of $\gamma$ in $\mathbb{R}^{n+1}$, we have

$$\kappa = \left\langle \nu \,\middle|\, \frac{d}{ds}\gamma_s \right\rangle = h_{ij} \gamma_s^i \gamma_s^j = 0 \,,$$

thus $\gamma$ is a straight line in $\mathbb{R}^{n+1}$.
Hence, all the $(n-m)$-dimensional affine subspace $p + S(p) \subset \mathbb{R}^{n+1}$ is contained in $M$, where we set $S(p) = \langle v_1(p), \dots, v_{n-m}(p)\rangle \subset \mathbb{R}^{n+1}$.
Let now $\sigma(s)$ be a geodesic from $p$ to another point $q$ parametrized by arclength and extend by parallel transport the vectors $v_k$ along $\sigma$, then

$$\nabla_s (h_{ij} v_k^j) = \mathrm{H}^{-1} \langle \nabla \mathrm{H} \,|\, \sigma_s\rangle h_{ij} v_k^j$$

and again by Gronwall's lemma it follows that $h_{ij}v_k^j(s) =$ for every $s \in \mathbb{R}$, in particular $v_k(q)$ is contained in the kernel of A at $q \in M$. This argument clearly shows that the kernel $S(p)$ of A has constant dimension $n - m$ with $0 < m < n$ (as A is never zero) at every point $p \in M$ and all the affine $(n-m)$-dimensional subspaces $p + S(p) \subset \mathbb{R}^{n+1}$ are contained in $M$.

Moreover, as $h_{ij}v_k^j = 0$ along the geodesic $\sigma$, denoting by $\nabla^{\mathbb{R}^{n+1}}$ the covariant derivative of $\mathbb{R}^{n+1}$ we have

$$\nabla_s^{\mathbb{R}^{n+1}} v_k = \nabla_s v_k + h_{ij} v_k^j \sigma_s^i \nu = 0\,,$$

so the extended vectors $v_k$ are constant in $\mathbb{R}^{n+1}$, which means that the parallel extension is independent of the geodesic $\sigma$, that the subspaces $S(p)$ are all a common $(n-m)$-dimensional vector subspace of $\mathbb{R}^{n+1}$ that we denote by $S$ and that $M = M + S \subset \mathbb{R}^{n+1}$.

By Sard's theorem, there exists a vector $y \in S$ such that $N = M \cap (y + S^\perp)$ is a smooth, complete $m$-dimensional submanifold of $\mathbb{R}^{n+1}$; then as $M = M + S$ it is easy to see that $M = N \times S$, but this implies that $L = S^\perp \cap M$ is a smooth, complete $m$-dimensional submanifold of $S^\perp = \mathbb{R}^{m+1}$ with $M = L \times S$.

Moreover, as $S$ is in the tangent space to every point of $L$, the normal $\nu$ to $M$ at a point of $L$ stays in $S^\perp$ so it must coincide with the normal $\nu^L$ to $L$ in $S^\perp$, then a simple computation shows that the mean curvature of $M$ at the points of $L$ is equal to the mean curvature $\mathrm{H}^L$ of $L$ as a hypersurface of $S^\perp = \mathbb{R}^{m+1}$. This shows that $L$ is a hypersurface in $\mathbb{R}^{m+1}$ satisfying $\mathrm{H}^L + \langle z \,|\, \nu^L \rangle = 0$ for every $z \in \mathbb{R}^{m+1}$.

Finally, as by construction the second fundamental form of $L$ has empty kernel, by the previous discussion we have $L = \mathbb{S}^m(\sqrt{m})$ and $M = \mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and we are done. $\square$

*Remark* 3.4.2. Notice that it follows that all the possible blow-up limits are convex. It is a very important fact, proved by Huisken and Sinestrari [73] (see also White [122]), that the same conclusion also holds at a type II singularity of the flow of a mean convex hypersurface (Theorem 4.2.1 and Proposition 4.2.3 in the next chapter).

*Remark* 3.4.3. Actually, Abresch and Langer in [1] (and also Epstein and Weinstein in [40]) classify *all* the closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$ (also the curves with self-intersections), see Appendix E for more details.

We underline that, even if the techniques are elementary, the proof of such a classification result is definitely nontrivial.

The result in the embedded case in these papers is a consequence of the general classification theorem. To the author's knowledge, the "shortcut" presented in the proof above is due to Chou and Zhu [26, Proposition 2.3].

We mention that recently Colding and Minicozzi in [29] proved this classification result assuming only a polynomial volume growth, without any bound on the second fundamental form A.

In dimension $n \geq 2$, without the assumption $\mathrm{H} > 0$ the conclusion is not true; an example is the Angenent torus in [17]. The following higher-dimensional analogy of Abresch and Langer's result is an open question.

**Open Problem 3.4.4.** When $n \geq 2$, is any smooth embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ isometric to a sphere?

Since by Corollary 3.3.7 the unit multiplicity hyperplanes are excluded as blow-up limits at a singular point, we have the following conclusion.

**Theorem 3.4.5.** *Let the compact, initial hypersurface be embedded and with* $\mathrm{H} \geq 0$*. Then, every limit hypersurface obtained by rescaling around a type I singular point, up to a rotation in* $\mathbb{R}^{n+1}$*, must be either the sphere* $\mathbb{S}^n(\sqrt{n})$ *or one of the cylinders* $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.

We discuss now what are the possible values of the limit heat density function, following Stone [115]. As $\Theta(p)$ is the value of the Huisken's functional on any limit of rescaled hypersurfaces and since these latter are "finite", we have that the possible values of $\Theta(p)$ are 1 in the case of a hyperplane and

$$\Theta^{n,m} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}} e^{-\frac{|x|^2}{2}} \, d\mathcal{H}^n$$

for $m \in \{1, \dots, n\}$.
A straightforward computation gives, for $m > 0$,

$$\Theta^{n,m} = \left(\frac{m}{2\pi e}\right)^{m/2} \omega_m$$

where $\omega_m$ denotes the volume of the unit $m$-sphere.
Notice that $\Theta^{n,m}$ does not depend on $n$ so we can simply write $\Theta^m = \Theta^{n,m}$.

**Lemma 3.4.6 (Stone [115]).** *The values of* $\Theta^m$ *are all distinct and larger than* 1 *for* $m > 0$*. Indeed the numbers* $\{\Theta^m \,|\, m = 1, 2, \dots\}$ *form a strictly decreasing sequence in* $m \in \mathbb{N}$*, with* $\Theta^m \searrow \sqrt{2}$ *as* $m \to \infty$.

By all this discussion we conclude that the "shape" of the limit hypersurfaces arising from a blow-up at a type I singularity of mean curvature flow of a compact, embedded, mean convex hypersurface, is classified by the value of the limit heat density function at the blow-up point.

Proposition 3.4.1 solves Problem 3.2.12 in the subclass of the embedded, limit hypersurfaces with $\mathrm{H} \geq 0$. A consequence is a positive answer to Problem 3.3.3, indeed, if the limit of rescalings around a nonspecial singular point is an embedded hypersurface with at least one point with $\mathrm{A} = 0$, the only possibility is then a single hyperplane with unit multiplicity by the classification result.
Moreover, combining such conclusion with Proposition 3.3.4, also Problem 3.3.6 has a full answer in this subclass, in its stronger form (see the discussion immediately after the problem).

**Proposition 3.4.7.** Every *singular point of a type I singularity of the motion by mean curvature of a compact, embedded initial hypersurface with* $\mathrm{H} \geq 0$ *is a special singular point.*

*Remark* 3.4.8. We underline that it follows that $\Sigma > 1$ without appealing to White's theorem, indeed, by this proposition there must exist at least one special singular point $p \in M$, hence $\Theta(p) > 1$ and $\Sigma \geq \Theta(p) > 1$.
This fact clearly allows also the conclusion of the blow-up procedure described in Remark 3.3.9 without White's theorem.

As a consequence, if the flow develops a type I singularity and a blow-up limit is a sphere (or a circle for curves), the flow is smooth till the hypersurface shrinks to a point, becoming asymptotically spherical.
This also implies that at some time the hypersurface has become convex.

Actually, more in general, the following pair of theorems describe the flow of convex curves and hypersurfaces.

**Theorem 3.4.9 (Gage and Hamilton [46, 47, 48]).** *Under the curvature flow a convex closed curve in* $\mathbb{R}^2$ *smoothly shrinks to a point in finite time. Rescaling in order to keep the length constant, it converges to a circle in* $C^\infty$.

**Theorem 3.4.10 (Huisken [65]).** *Under the mean curvature flow a compact and convex hypersurface in* $\mathbb{R}^{n+1}$ *with* $n \geq 2$ *smoothly shrinks to a point in finite time. Rescaling in order to keep the* Area *constant, it converges to a sphere in* $C^\infty$.

*Remark* 3.4.11. The theorem for curves is not merely a consequence of the general result. The proof in dimension $n \geq 2$ does not work in the one-dimensional case.

Actually, the $C^\infty$-convergence to a circle or to a sphere is exponential.

At the end of Section 4.1 of the next chapter, we will show a line of proof of Theorem 3.4.10 by Hamilton in [60], different from the original one. Another proof was also given by Andrews in [9], analyzing the behavior of the eigenvalues of the second fundamental form close to the singular time.

Theorem 3.4.9 will follow from the strong fact that a simple closed curve in $\mathbb{R}^2$ cannot develop type II singularities at all.

The last point missing in all this story, even in the mean convex case when $n \geq 2$, is a full answer to Problem 3.2.11. We concluded that any blow-up limit gives the same value of the Huisken's functional, hence its "shape" is fixed: hyperplane, sphere or cylinder. If the limit is a sphere, the limit is unique and there is full convergence; if it is a hyperplane we already had such a conclusion by White's Theorem 3.2.22. But, if the limit is a cylinder, its axis could possibly change, depending on the choice of the converging sequence. Clearly, in the case of curves Problem 3.2.11 is solved affirmatively as there are no "cylinders".

If the initial hypersurface is only immersed, the conclusion of the classification theorem, Proposition 3.4.1, still holds, allowing the possibility that $M$ is the *union* of a finite family of hypersurfaces among the hyperplanes through the origin, the spheres $\mathbb{S}^n(\sqrt{n})$ and the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ (with possible different axes).

In the one-dimensional case one has to enlarge the possible curves in the conclusion of such theorem to include also all the family of the so-called Abresch–Langer curves, described in [1] (see Appendix E).

Anyway, the possible existence of blow-up limits which are hyperplanes with multiplicity larger than one prevents the application of the argument leading to Proposition 3.4.7. Actually, to the author's knowledge, in the immersed-only case there is not a general procedure to exhibit a limit of rescaled hypersurfaces which is homothetic and nonflat at a type I singularity.

## 3.5 Embedded Closed Curves in the Plane

The case of an embedded, closed curve $\gamma$ in $\mathbb{R}^2$ is special, indeed the classification theorem, Proposition 3.4.1, holds without *a priori* assumptions on the curvature. So there are only two possible limits of rescaled curves without self-intersections, either a line through the origin or the circle $\mathbb{S}^1$. This gives immediately a general positive answer to Problems 3.2.12 and 3.3.3 and implies as before that every singular point is a special singular point.
As we already said, in this very special case, also Problem 3.2.11 is solved affirmatively; the limit is always unique.

Arguing as in the previous section, we then have the following conclusion.

**Theorem 3.5.1.** *Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve, then every curve obtained by limit of rescalings around a type I singular point of its motion by curvature is the circle $\mathbb{S}^1$.*
*As a consequence, if a simple closed curve is developing a type I singularity, at some time the curve becomes convex and shrinks to a point, becoming asymptotically circular at the singular time.*

We mention here that an extensive and deep analysis of the behavior of general curves moving by curvature (even when the ambient is a generic surface different from $\mathbb{R}^2$) is provided by the pair of papers by Angenent [14, 16] (see also the discussion in [17]).

# Chapter 4

# Type II Singularities

We assume now that we are in the type II singularity case, that is,

$$\limsup_{t\to T} \max_{p\in M} |\mathrm{A}(p,t)|\sqrt{T-t} = +\infty$$

for the mean curvature flow of a compact hypersurface $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ in its maximal interval of existence.

A good question is actually whether type II singularities exist.

An example is given by a closed, symmetric, self-intersecting curve with the shape of a symmetric "eight" figure in the plane, which has zero *rotation number*. Pushing a little the analysis of the previous chapter and keeping into account the symmetries of the curve, if the curve develops a type I singularity, we can produce a nonflat blow-up limit which is homothetic and nonflat. Then such a limit must be a circle or one of Abresch–Langer's curves. In both cases, the limit would be a compact closed curve and by the smooth convergence, the rotation number would still be zero. Hence, the circle has to be excluded and the contradiction is given by the fact that there are no Abresch–Langer curves with zero rotation number. Hence, type I singularities do not describe all the possible ones.

Another example is given by a cardioid-like curve in the plane with a very small loop, hence high curvature: one can correctly guess that at some time the loop shrinks while the rest of the curve remains smooth and a cusp develops. Such a singularity is of type II, since if we have a Type I singularity we would get an Abresch–Langer curve as a blow-up limit and this implies, as these latter are compact, that the entire curve has vanished in a single point (see the analysis in [15] and also [14, 16]).

As we will see in Theorem 4.5.5 that embedded curves do not develop type II singularities, one could reasonably conjecture that also for embedded hypersurfaces (at least in low dimension) all the singularities are of type I. Unfortunately, this is not true even if the dimension is only two, indeed, the following example excludes such a good behavior.

C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progress in Mathematics 290,
DOI 10.1007/978-3-0348-0145-4_4, 

*Example* (*The Degenerate Neckpinch*). For a given $\lambda > 0$, let us set

$$\phi_\lambda(x) = \sqrt{(1-x^2)(x^2+\lambda)}, \qquad -1 \leq x \leq 1.$$

For any $n \geq 2$, let $M^\lambda$ be the $n$-dimensional hypersurface in $\mathbb{R}^{n+1}$ obtained by rotation of the graph of $\phi_\lambda$ in $\mathbb{R}^2$. The hypersurface $M^\lambda$ looks like a dumbbell, where the parameter $\lambda$ measures the width of the central part. Then, it is possible to prove the following properties (see [4]):

1. if $\lambda$ is large enough, the hypersurface $M_t^\lambda$ eventually becomes convex and shrinks to a point in finite time;
2. if $\lambda$ is small enough, $M_t^\lambda$ exhibits a neckpinch singularity as in the case of the *standard neckpinch* (see Section 1.4);
3. there exists at least one intermediate value of $\lambda > 0$ such that $M_t^\lambda$ shrinks to a point in finite time, has positive mean curvature up to the singular time, but never becomes convex. The maximum of the curvature is attained at the two points where the surface meets the axis of rotation;
4. in this latter case the singularity is of type II, otherwise the blow-up at the singular time would give a sphere (for all $p \in M$ we would have $\widehat{p} = O \in \mathbb{R}^{n+1}$ hence, by estimate (3.2.2), any limit hypersurface is bounded). This is impossible as it would imply that the surface would have been convex at some time.

The flowing hypersurface at point (3) is called the *degenerate neckpinch* and was first conjectured by Hamilton for the Ricci flow [61, Section 3]. Intuitively speaking, it is a limiting case of the neckpinch where the cylinder in the middle and the two spheres on the sides shrink at the same time. One can also build the example in an asymmetric way, with only one of the two spheres shrinking simultaneously with the neck, while the other one remains nonsingular.

A sharp analysis of the singular behavior for a class of rotationally symmetric surfaces exhibiting a degenerate neckpinch has been done by Angenent and Velázquez in [19].

Another interesting example of singularity formation (a family of evolving tori, proposed by De Giorgi) was carefully studied by Soner and Souganidis in [110, Proposition 3] (see also the numerical analysis performed by Paolini and Verdi in [101, Section 7.5]).

## 4.1 Hamilton's Blow-up

In order to deal with the blow-up around type II singularities we need a new set of estimates which are actually independent of the type II hypothesis and scaling invariant (see [3] and [104]).

**Proposition 4.1.1.** *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p\in M} |\mathrm{A}(p,0)| \leq \Lambda < +\infty$. Then, there exists a time $\tau = \tau(\Lambda) > 0$ and constants $C_m = C_m(\Lambda)$, for every $m \in \mathbb{N}$ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m/t^m$ for every $p \in M$ and $t \in (0,\tau)$.*

*Proof.* We prove the claim by induction. By the evolution equation for $|\mathrm{A}|^2$,

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4,$$

we get

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,,$$

hence, there exists a time $\tau = \tau(\Lambda) > 0$ and a constant $C_0 = C_0(\Lambda)$ such that $|\mathrm{A}(p,t)|^2 \leq C_0$ for every $p \in M$ and $t \in [0,\tau)$. This is the case $m = 0$.
Recalling equation (2.3.4), setting $f = \sum_{k=0}^m |\nabla^k\mathrm{A}|^2\lambda_k t^k$ for some positive constants $\lambda_0,\dots,\lambda_m$ and assuming the inductive hypothesis $|\nabla^k\mathrm{A}(p,t)|^2 \leq C_k(\Lambda)/t^k$ for any $k \in \{0,\dots,m-1\}$, $p \in M$ and $t \in (0,\tau)$, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} f &= \frac{\partial}{\partial t}\sum_{k=0}^m |\nabla^k\mathrm{A}|^2\lambda_k t^k \\
&= \sum_{k=1}^m |\nabla^k\mathrm{A}|^2 k\lambda_k t^{k-1} \\
&\quad + \sum_{k=0}^m \lambda_k t^k\Big(\Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}\Big) \\
&\leq \Delta f + \sum_{k=1}^m |\nabla^k\mathrm{A}|^2(k\lambda_k - 2\lambda_{k-1})t^{k-1} - 2|\nabla^{m+1}\mathrm{A}|^2\lambda_m t^m \\
&\quad + \sum_{k=0}^m \lambda_k t^k C(k) \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} |\nabla^p\mathrm{A}||\nabla^q\mathrm{A}||\nabla^r\mathrm{A}||\nabla^k\mathrm{A}| \\
&\leq \Delta f + \sum_{k=1}^m |\nabla^k\mathrm{A}|^2(k\lambda_k - 2\lambda_{k-1})t^{k-1} + \sum_{k=0}^{m-1}\lambda_k C(k)\sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} C_pC_qC_rC_k \\
&\quad + \lambda_m t^{m/2}C(m)\Big(\sum_{p+q+r=m\,|\,p,q,r<m} C_pC_qC_r\Big)|\nabla^m\mathrm{A}| + \lambda_m t^m C(m)|\mathrm{A}|^2|\nabla^m\mathrm{A}|^2 \\
&\leq \Delta f + \sum_{k=1}^m |\nabla^k\mathrm{A}|^2(k\lambda_k - 2\lambda_{k-1})t^{k-1} + C\lambda_m t^m|\nabla^m\mathrm{A}|^2 + D
\end{aligned}$$

where in the last passage we applied Peter–Paul inequality. If we choose now inductively positive constants $\lambda_1,\dots,\lambda_m$ such that $\lambda_k = 2\lambda_{k-1}/k$ starting with

$\lambda_0 = 1$ (easily $\lambda_k = 2^k/k!$), we have

$$\frac{\partial}{\partial t} f \leq \Delta f + C\lambda_m t^m |\nabla^m \mathrm{A}|^2 + D \leq \Delta f + Cf + D\,,$$

for every $p \in M$ and $t \in (0,\tau)$, where the constants $C$ and $D$ depend only on $m$ and $\Lambda$, by the inductive hypothesis. Notice that the inequality holds also at $t = 0$ as the function $f$ is smooth on $M \times [0,\tau)$.
This differential inequality, by the maximum principle, then implies that $f_{\max}(t)$ is bounded in the interval $[0,\tau)$ by some constant $C$ depending only on $m$, $\Lambda$ and $f_{\max}(0) = |\mathrm{A}|^2_{\max}(0) \leq \Lambda^2$, hence

$$t^m |\nabla^m \mathrm{A}(p,t)|^2 \leq f(t)/\lambda_m \leq C/\lambda_m = C_m$$

in the interval $t \in [0,\tau)$; then we are done as $C_m = C_m(\Lambda)$. $\square$

The following corollary is an easy consequence.

**Corollary 4.1.2.** *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p\in M} |\mathrm{A}(p,0)| \leq \Lambda < +\infty$. Then, there exists a value $\tau = \tau(\Lambda) > 0$ and constants $C_m$ for every $m \in \mathbb{N}$, depending only on $\Lambda$, such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m$ for every $p \in M$ and $t \in (\tau/2,\tau)$.*
*For instance, one can choose $\tau = 1/(4\Lambda^2)$.*

*Proof.* Only the last claim needs an explanation, it follows by integrating the differential inequality

$$\frac{\partial}{\partial t} |\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,. \qquad \square$$

*Remark* 4.1.3. These estimates provide another proof of Proposition 2.4.8, moreover they can replace the estimates of Proposition 3.2.9 in the proof of Proposition 3.2.10.

We describe now Hamilton's procedure to get a blow-up flow at a type II singularity of the mean curvature flow of a compact hypersurface at time $T > 0$.

Let us choose a sequence of times $t_k \in [0, T - 1/k]$ and points $p_k \in M$ such that

$$|\mathrm{A}(p_k,t_k)|^2 (T - 1/k - t_k) = \max_{\substack{t\in[0,T-1/k]\\ p\in M}} |\mathrm{A}(p,t)|^2 (T - 1/k - t)\,. \tag{4.1.1}$$

This maximum goes to $+\infty$ as $k \to \infty$; indeed, if it is bounded by some constant $C$ on a subsequence $k_i \to \infty$, for every $t \in [0,T)$ definitely we have $t \in [0, T - 1/k_i]$ and

$$|\mathrm{A}(p,t)|^2 (T - t) = \lim_{i\to\infty} |\mathrm{A}(p,t)|^2 (T - 1/k_i - t) \leq C$$

for every $p \in M$. This is in contradiction with the type II condition

$$\limsup_{t\to T} \max_{p\in M} |\mathrm{A}(p,t)| \sqrt{T-t} = +\infty\,.$$

This fact also forces the sequence $t_k$ to converge to $T$ as $k \to \infty$. If $t_{k_i}$ is a subsequence not converging to $T$, we would have that the sequence $|\mathrm{A}(p_{k_i}, t_{k_i})|^2$

is bounded, hence also

$$\max_{\substack{t\in[0,T-1/k_i]\\ p\in M}} |\mathrm{A}(p,t)|^2(T-1/k_i-t)$$

would be bounded.
Thus, we can choose an increasing (not relabelled) subsequence $t_k$ converging to $T$, such that $|\mathrm{A}(p_k,t_k)|$ goes monotonically to $+\infty$ and

$$|\mathrm{A}(p_k,t_k)|^2 t_k \to +\infty\,, \qquad |\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)\to+\infty\,,$$

Moreover, we can also assume that $p_k\to p$ for some $p\in M$.
We rescale now *the flow* as follows: let $\varphi_k : M\times I_k\to\mathbb{R}^{n+1}$, where

$$I_k=[-|\mathrm{A}(p_k,t_k)|^2t_k, |\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)]\,,$$

be the evolution given by

$$\varphi_k(p,s)=|\mathrm{A}(p_k,t_k)|[\varphi(p,s/|\mathrm{A}(p_k,t_k)|^2+t_k)-\varphi(p_k,t_k)]$$

and we set $M^k_s=\varphi_k(M,s)$ and $\mathrm{A}_k$ the second fundamental form of the flowing hypersurfaces $\varphi_k$.
It is easy to check that this is a parabolic rescaling hence, every $\varphi_k$ is still a mean curvature flow, moreover the following properties hold:

- $\varphi_k(p_k,0)=0\in\mathbb{R}^{n+1}$ and $|\mathrm{A}_k(p_k,0)|=1$,
- for every $\varepsilon>0$ and $\omega>0$ there exists $\overline{k}\in\mathbb{N}$ such that

$$\max_{p\in M}|\mathrm{A}_k(p,s)|\le 1+\varepsilon \tag{4.1.2}$$

  for every $k\ge\overline{k}$ and $s\in[-|\mathrm{A}(p_k,t_k)|^2t_k,\omega]$.

Indeed, (the first point is immediate), by the choice of the minimizing pairs $(p_k,t_k)$ we get

$$\begin{aligned}|\mathrm{A}_k(p,s)|^2 &= |\mathrm{A}(p_k,t_k)|^{-2}|\mathrm{A}(p,s/|\mathrm{A}(p_k,t_k)|^2+t_k)|^2\\ &\le |\mathrm{A}(p_k,t_k)|^{-2}|\mathrm{A}(p_k,t_k)|^2\frac{T-1/k-t_k}{T-1/k-t_k-s/|\mathrm{A}(p_k,t_k)|^2}\\ &=\frac{|\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)}{|\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)-s}\,,\end{aligned}$$

if $s/|\mathrm{A}(p_k,t_t)|^2+t_k\in[0,T-1/k]$, that is, if $s\in I_k$. Then, assuming $s\le\omega$ and $k$ large enough, the claim follows as we know that $|\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)\to+\infty$.

This discussion implies that if we are able to take a (subsequential) limit of these *flows*, locally smoothly converging in every compact time interval, we would get a mean curvature flow such that the norm of the second fundamental form is uniformly bounded by one and the time interval of existence is the whole $\mathbb{R}$ as $\lim_{k\to\infty} I_k=(-\infty,+\infty)$.
This is ensured by the next proposition.

**Proposition 4.1.4.** *The family of flows $\varphi_k$ converges (up to a subsequence) in the $C^\infty_{\mathrm{loc}}$ topology to a nonempty, smooth evolution by mean curvature of complete hypersurfaces $M^\infty_s$ in the time interval $(-\infty, +\infty)$. Such a flow is called* eternal, *as a consequence it cannot contain compact hypersurfaces.*
*Moreover, the second fundamental form and all its covariant derivatives are uniformly bounded and $|\mathrm{A}_\infty|$ takes its absolute maximum, which is 1, at time $s = 0$ at the origin of $\mathbb{R}^{n+1}$, hence the limit flow is nonflat.*
*Finally, if the original initial hypersurface was embedded this limit flow consists of embedded hypersurfaces.*

*Proof.* By the previous discussion, in every bounded interval of time $[s_1, s_2]$ the evolutions $\varphi_k$ have definitely uniformly bounded curvature, precisely $|\mathrm{A}_k| \leq (1+\varepsilon)$, then for $\varepsilon << 1$ by Corollary 4.1.2 in every interval $[s_1 + 1/16, s_1 + 1/8]$ we have uniform estimates $|\nabla^m \mathrm{A}_k| \leq C_m$ with $C_m$ independent of $s_1$, for every $m \in \mathbb{N}$.

By means of the monotonicity formula we can have a uniform estimate on $\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R)$ as follows (we recall that $\widetilde{\mathcal{H}}^n$ is the $n$-dimensional Hausdorff measure counting multiplicities): we set $\mu^k_s$ to be the measure associated to the hypersurface $\varphi_k$ at time $s$ and $\mu_0$ the measure associated to the initial hypersurface $\varphi_0$, then

$$
\begin{aligned}
\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) &= \int_M \chi_{B_R}(y)\, d\mu^k_s(y) \\
&\leq \int_M \chi_{B_R}(y) e^{\frac{R^2 - |y|^2}{4}}\, d\mu^k_s(y) \\
&\leq e^{R^2/4} \int_M e^{-\frac{|y|^2}{4}}\, d\mu^k_s(y) \\
&= (4\pi)^{n/2} e^{R^2/4} \int_M \frac{e^{-\frac{|y|^2}{4(s+1-s)}}}{[4\pi(s+1-s)]^{n/2}}\, d\mu^k_s(y) \\
&\leq C(R) \int_M \frac{e^{-\frac{|y|^2}{4(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)}}}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)]^{n/2}}\, d\mu^k_{-|\mathrm{A}(p_k,t_k)|^2 t_k}(y) \\
&= C(R) \int_M \frac{|\mathrm{A}(p_k,t_k)|^n e^{-\frac{|x-\varphi(p_k,t_k)|^2 |\mathrm{A}(p_k,t_k)|^2}{4(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)}}}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)]^{n/2}}\, d\mu_0(x) \\
&\leq C(R) \int_M \frac{|\mathrm{A}(p_k,t_k)|^n}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)]^{n/2}}\, d\mu_0(x) \\
&\leq C(R) \mathrm{Area}(\varphi_0) \frac{|\mathrm{A}(p_k,t_k)|^n}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2 t_k)]^{n/2}}\,,
\end{aligned}
$$

hence, if $s$ stays in a bounded interval $[s_1, s_2] \subset \mathbb{R}$, we have

$$
\limsup_{k\to\infty} \widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) \leq C(R) \frac{\mathrm{Area}(\varphi_0)}{[4\pi T]^{n/2}} = C(R, \varphi_0)\,.
$$

This implies that

$$\widetilde{\mathcal{H}}^n(\varphi_k(M,s)\cap B_R)\leq C(R,\varphi_0,s_1,s_2)$$

uniformly in $s\in[s_1,s_2]$ and where the constant $C$ is independent of $k\in\mathbb{N}$.
Then we use the same argument of Proposition 3.2.10, but applied to *flows*, that is, we consider the time-tracks of the flows $\varphi_k$ as hypersurfaces $\widetilde{\varphi}_k : M\times I_k\to\mathbb{R}^{n+1}\times\mathbb{R}=\mathbb{R}^{n+2}$ defined by $\widetilde{\varphi}_k(p,s)=(\varphi_k(p,s),s)$ and we reparametrize them locally as graphs of smooth functions.
Reasoning as in the proof of Proposition 2.4.9, the estimates on the space covariant derivatives of $\mathrm{A}_k$ imply uniform locally estimates on space and also time derivatives (using the evolution equation) of the representing functions, so up to a subsequence we can get locally a limit smooth mean curvature flow. By a diagonal argument, we have the existence of a limit flow (follow the proof of Proposition 3.2.10).
The claimed properties of such limit flow are immediate by the above discussion and by the fact that any compact hypersurface cannot give rise to an eternal flow by Corollary 2.2.5. The only point requiring a justification is the embeddedness, if the initial hypersurface is embedded.
In this case, by Proposition 2.2.7, all the hypersurfaces in the flows $\varphi_k$ are embedded at every time, then the only possibility for $M^\infty_s$ not to be embedded is if two or more of its regions "touch" each other at some point $y\in\mathbb{R}^{n+1}$ with a common tangent hyperplane.
We define the monotone nondecreasing function $G(t)=\max_{\substack{s\in[0,t]\\ p\in M}}|\mathrm{A}(p,s)|$ and we choose a smooth, monotone nondecreasing function $K:[0,T)\to\mathbb{R}^+$ such that $G(t)\leq K(t)\leq 2G(t)$ for every $t\in[0,T)$.
Then, we consider the following open set $\Omega_\varepsilon\subset M\times M\times[0,T)$ given by

$$\{(p,q,t)\,|\,d_{g(t)}(p,q)\leq\varepsilon/K(t)\},$$

where $d_{g(t)}$ is the geodesic distance in the Riemannian manifold $(M,g(t))$. Let

$$B_\varepsilon=\inf_{\partial\Omega_\varepsilon}|\varphi(p,t)-\varphi(q,t)|K(t)$$

and suppose that $B_\varepsilon=0$ for some $\varepsilon>0$. This means that there exists a sequence of times $t_i\nearrow T$ and points $p_i$, $q_i$ with $d_{g(t_i)}(p_i,q_i)=\varepsilon/K(t_i)$ and $|\varphi(p_i,t_i)-\varphi(q_i,t_i)|K(t_i)\to 0$, that is, $|\widetilde{\varphi}_i(p_i)-\widetilde{\varphi}_i(q_i)|\to 0$ and $d_{\widetilde{g}(s_i)}(p_i,q_i)=\varepsilon$, where $\widetilde{\varphi}_i$ is the rescaling of the hypersurface $\varphi_{t_i}$ around the point $\varphi(p_i,t_i)$ by the dilation factor $K(t_i)\geq G(t_i)$.
As the curvatures $\mathrm{A}_i$ of these rescaled hypersurfaces $\widetilde{\varphi}_i$ satisfy

$$|\mathrm{A}_i(p)|=|\mathrm{A}(p,t_i)|/K(t_i)\leq|\mathrm{A}(p,t_i)|/G(t_i)\leq 1\,,$$

reasoning as in the proof of Proposition 3.2.10, we have a contradiction if $\varepsilon>0$ is small enough.

Now, fixing $\varepsilon > 0$ such that the above constant $B_\varepsilon$ is positive and looking at the function

$$L(p,q,t) = |\varphi(p,t) - \varphi(q,t)| K(t)$$

on $\complement\Omega_\varepsilon \subset M \times M \times [0,T)$, we have that if the minimum of $L$ at any time $t$ (which is positive as the hypersurfaces are embedded) is lower than $B_\varepsilon$, then such a minimum is not taken on the boundary of the set but in its interior, say at a pair $(p,q)$. Then, we compute at the point $(p,q,t)$,

$$\begin{aligned}\frac{\partial L(p,q,t)}{\partial t} &= K(t)\frac{\partial}{\partial t}|\varphi(p,t) - \varphi(q,t)| + |\varphi(p,t) - \varphi(q,t)|K'(t)\\ &\geq K(t)\frac{\partial}{\partial t}|\varphi(p,t) - \varphi(q,t)|\end{aligned}$$

and a geometric argument analogous to the one in the proof of Proposition 2.2.7 shows that this last partial derivative is nonnegative (when it exists, almost everywhere). Then, by means of the maximum principle (Hamilton's trick, Lemma 2.1.3) we conclude that when the minimum of $L$ at time $t$ is lower than $B_\varepsilon$ it is nondecreasing.

Hence, there is a positive lower bound $C_\varepsilon$ on

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)| K(t)\,,$$

consequently,

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p,t) - \varphi(q,t)| G(t) \geq C_\varepsilon/2 > 0\,.$$

Now notice that, for the pairs $(p_k, t_k)$ coming from formula (4.1.1) we have $|\mathrm{A}(p_k,t_k)| = G(t_k)$, otherwise there would exist a time $t < t_k$ with

$$\max_{p\in M} |\mathrm{A}(p,t)| > |\mathrm{A}(p_k,t_k)|$$

which is in contradiction with the maximum in the right-hand side of equation (4.1.1).

As $|\mathrm{A}(p_k,t_k)| \geq G(t)$ for every $t \leq t_k$, fixed $\omega, \delta > 0$, by inequality (4.1.2) we have definitely $\max_{p\in M} |\mathrm{A}_k(p,s)| \leq (1+\delta)$ for every $s \leq \omega$, hence

$$\begin{aligned}G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k) &= \max_{\substack{r\leq s\\ p\in M}} |\mathrm{A}(p, r/|\mathrm{A}(p_k,t_k)|^2 + t_k)| \\ &\leq \max_{\substack{r\leq \omega\\ p\in M}} |\mathrm{A}_k(p,r)||\mathrm{A}(p_k,t_k)| \\ &\leq (1+\delta)|\mathrm{A}(p_k,t_k)|\,.\end{aligned} \tag{4.1.3}$$

If $s \in [0, \omega]$ and $d_{g_k(s)}(p, q) > \varepsilon$, definitely

$$\begin{aligned} d_{g(s/|\mathrm{A}(p_k,t_k)|^2+t_k)}(p, q) &= d_{g_k(s)}(p, q)/|\mathrm{A}(p_k, t_k)| \\ &= d_{g_k(s)}(p, q)/G(t_k) \\ &\geq \frac{\varepsilon}{G(s/|\mathrm{A}(p_k, t_k)|^2 + t_k)} \\ &\geq \frac{\varepsilon}{K(s/|\mathrm{A}(p_k, t_k)|^2 + t_k)} \end{aligned}$$

hence, $(p, q, s/|\mathrm{A}(p_k, t_k)|^2 + t_k) \in \complement\Omega_\varepsilon$.
If instead $s \leq 0$, we define $L(s) = \sup_{M_s^\infty} |\mathrm{A}_\infty| \leq 1$ and we see that if $L(s) = 0$ for some $s \leq 0$ then $M_s^\infty$ is a hyperplane and the limit flow is flat till $s = 0$ (by uniqueness of the flow as $\mathrm{A}_\infty$ is bounded, see Remark 1.5.4), which is impossible as $|\mathrm{A}_\infty(0, 0)| = 1$, hence $L(s) > 0$. Then, for every $s \leq 0$ we must have definitely

$$G(s/|\mathrm{A}(p_k, t_k)|^2 + t_k)/|\mathrm{A}(p_k, t_k)| \geq L(s)/2$$

and if $d_{g_k(s)}(p, q) > 2\varepsilon/L(s)$,

$$\begin{aligned} d_{g(s/|\mathrm{A}(p_k,t_k)|^2+t_k)}(p, q) &= d_{g_k(s)}(p, q)/|\mathrm{A}(p_k, t_k)| \\ &> \frac{2\varepsilon}{|\mathrm{A}(p_k, t_k)|L(s)} \\ &\geq \frac{\varepsilon}{G(s/|\mathrm{A}(p_k, t_k)|^2 + t_k)} \\ &\geq \frac{\varepsilon}{K(s/|\mathrm{A}(p_k, t_k)|^2 + t_k)} \end{aligned}$$

hence, also in this case $(p, q, s/|\mathrm{A}(p_k, t_k)|^2 + t_k) \in \complement\Omega_\varepsilon$.
Then in both cases, if $d_{g_k(s)}(p, q) > \min\{\varepsilon, 2\varepsilon/L(s)\} = \varepsilon > 0$ (notice that $\varepsilon < 2\varepsilon/L(s)$ as $L(s) \leq 1$),

$$\Big|\varphi(p, s/|\mathrm{A}(p_k,t_k)|^2 + t_k) - \varphi(q, s/|\mathrm{A}(p_k,t_k)|^2 + t_k)\Big| G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k) \geq C_\varepsilon/2 > 0$$

and by inequality (4.1.3) it follows that definitely

$$\begin{aligned} |\varphi_k(p,s) - \varphi_k(q,s)| &= |\varphi(p, s/|\mathrm{A}(p_k,t_k)|^2 + t_k) - \varphi(q, s/|\mathrm{A}(p_k,t_k)|^2 + t_k)|\,|\mathrm{A}(p_k,t_k)| \\ &\geq \frac{C_\varepsilon |\mathrm{A}(p_k,t_k)|}{2G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k)} \\ &\geq \frac{C_\varepsilon}{2(1+\delta)}. \end{aligned}$$

As $\omega$ and $\delta$ were arbitrary and the convergence is smooth, this conclusion passes to all the limit hypersurfaces $M_s^\infty$, for every $s \in \mathbb{R}$. That is, if a couple of points of $M_s^\infty$ has intrinsic distance larger than $\varepsilon > 0$, their extrinsic distance is bounded from below by some uniform positive constant. If $\varepsilon > 0$ is then chosen small enough such that any hypersurface with $|\mathrm{A}| \leq 1$ (like every $M_s^\infty$) is an embedding when it is restricted to any intrinsic ball of radius smaller than $\varepsilon$, we are done. The hypersurfaces $M_s^\infty$ cannot have self-intersections for every $s \in \mathbb{R}$, hence they are all embedded. □

**Exercise 4.1.5.** This blow-up procedure can be applied also at a type I singularity. There are some differences and the sequence $t_k$ must be chosen in order that $t_k \to T$, since it is not a consequence of the construction.
The limit mean curvature flow that one obtains is no more eternal but only *ancient*, that is, defined on some interval $(-\infty, \Omega)$ with $\Omega > 0$, and $|\mathrm{A}_\infty| \leq 1$ holds only on $(\infty, 0]$.
It is an open problem if this limit flow is actually homothetically shrinking, in general.

The analysis of singularities in the type II case is then reduced to classify these eternal flows with bounded curvature (and its covariant derivatives) with the extra property that the norm of the second fundamental form takes its maximum, equal to one, at some point in space and time.

Examples of this class are the *translating* mean curvature flows (with bounded second fundamental form and $|\mathrm{A}|$ achieving its maximum), that is, hypersurfaces $M \subset \mathbb{R}^{n+1}$ such that during the motion do not change their shape but simply move in a fixed direction with constant velocity. We have seen in Proposition 1.4.2 that this condition is equivalent to the existence of a vector $v \in \mathbb{R}^{n+1}$ such that $\mathrm{H}(p) = \langle v \,|\, \nu(p) \rangle$ at every point $p \in M$. Clearly, by comparison with spheres, these hypersurfaces cannot be compact.

**Open Problem 4.1.6.** Classify all the eternal mean curvature flows of complete, connected, hypersurfaces in $\mathbb{R}^{n+1}$ such that A and its covariant derivatives are uniformly bounded and $|\mathrm{A}|$ takes its maximum at some point in space-time. The same problem assuming embeddedness or supposing that the flow comes from Hamilton's blow-up procedure.

Another problem is the analogous classification for *ancient* complete flows with bounded curvature at every fixed time (see the discussion in [121, page 536]). For closed convex curves, this problem has been solved by Daskalopoulos, Hamilton and Sesum [31]. The higher-dimensional case was recently studied by Brendle, Huisken and Sinestrari. Finally, the same questions can be asked also for the *immortal* flows, that is, defined on $[0, +\infty)$.

In view of the results of the next section, we also state the following.

**Open Problem 4.1.7.** All the eternal mean curvature flows of complete hypersurfaces in $\mathbb{R}^{n+1}$ coming from Hamilton's blow-up procedure are translating flows? At least if they are embedded?

These problems are difficult in general, but as in the type I singularity case, if the evolving hypersurfaces are mean convex ($\mathrm{H} \geq 0$) or if we are dealing with curves in the plane, they have a positive answer. This will be the subject of the next sections.

We underline that the "bad blow-up rate" is an obstacle to the use of Huisken's monotonicity formula in the context of type II singularities, an exception will be discussed in Section 4.5.1.

We conclude this section by giving Hamilton's line of proof of Theorem 3.4.10, which is different from the original one.

*Proof of Theorem* 3.4.10. Let $T$ be the maximal time of smooth existence of the mean curvature flow of an $n$-dimensional convex hypersurface. By the results of Section 2.5, in particular Proposition 2.5.8, we have that after any positive time $\mathrm{H} > 0$ and there exists a positive constant $\alpha$, independent of time, such that $\mathrm{A} \geq \alpha \mathrm{H} g$ as forms.

If at time $T$ we have a type II singularity, we get an unbounded, eternal convex blow-up limit flow with $\mathrm{H} \geq 0$, using Hamilton's procedure. By the strong maximum principle, actually $\mathrm{H} > 0$ for every time (otherwise $\mathrm{H} \equiv 0$ everywhere, but this and the convexity would imply that the limit flow is simply a fixed hyperplane) and the condition $\mathrm{A} \geq \alpha \mathrm{H} g$ passes to the limit. Then, by the following theorem of Hamilton [60], all the hypersurfaces of the limit flow are compact, in contradiction with the unboundedness, hence type II singularities cannot develop.

**Theorem 4.1.8.** *Let $M$ be a smooth, complete, strictly convex, $n$-dimensional hypersurface in the Euclidean space, with $n \geq 2$. Suppose that for some $\alpha > 0$ its second fundamental form is $\alpha$-pinched in the sense that $\mathrm{A} \geq \alpha \mathrm{H} g$, where $g$ is the induced metric and* H *its mean curvature. Then $M$ is compact.*

Dealing with type I singularities, any blow-up limit is embedded, strictly convex and compact, again by this theorem. Hence, by Theorem 3.4.5 it can be only the sphere $\mathbb{S}^n(\sqrt{n})$.

This implies that the full sequence of rescaled hypersurfaces converges in $C^\infty$ to such sphere.

Finally, as the blow-up limit is unique and compact, the original hypersurface shrinks to a point in finite time. □

## 4.2 Hypersurfaces with Nonnegative Mean Curvature

We shall now consider the formation of type II singularities for hypersurfaces which are mean convex, that is, with nonnegative mean curvature everywhere.

An important result for the analysis of singularities of mean convex hypersurfaces is the following estimate on the elementary symmetric polynomials of the curvatures $\mathrm{S}_k$ proved in [73], which holds in general for any mean curvature flow.

**Theorem 4.2.1 (Huisken–Sinestrari [73]).** *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact, mean convex, immersed hypersurface, for $n \geq 2$.*

*Then, for any $\eta > 0$ there exists a positive constant $C = C(\eta, \varphi_0)$ such that $\mathrm{S}_k \geq -\eta \mathrm{H}^k - C$ for any $k = 2, \dots, n$ at every point of $M$ and $t \in [0, T)$.*

This estimate easily implies the following one, which has a more immediate interpretation.

**Corollary 4.2.2.** *Under the same hypotheses of the previous theorem, for any $\eta > 0$ there exists a positive constant $C = C(\eta, \varphi_0)$ such that $\lambda^{\min} \geq -\eta\mathrm{H} - C$ at every point of $M$ and $t \in [0, T)$, where $\lambda^{\min}$ is the smallest eigenvalue of the second fundamental form.*

The interest in the above estimates lies in the fact that $\eta$ can be chosen arbitrarily small and $C$ is a constant not depending on the curvatures and time. Thus, roughly speaking, we see that the negative curvatures become negligible with respect to the others when the singular time is reached, as H is going to $+\infty$. This implies that the second fundamental form of the hypersurface becomes asymptotically nonnegative definite at a singularity.

Let us observe that these results cannot be valid for general hypersurfaces, even in low dimension. Indeed, Angenent's homothetically shrinking torus in [17] has a behavior which is incompatible with these convexity estimates.

**Proposition 4.2.3.** *If $n \geq 2$ and the initial hypersurface is mean convex, the limit flow $M_s^\infty$ obtained by Hamilton's procedure, described in the previous section, consists of convex hypersurfaces.*

*Proof.* First, since we are taking the limit of hypersurfaces with $\mathrm{H} \geq 0$ the limit also is mean convex. By the strong maximum principle applied to the evolution equation for the mean curvature of the limit flow $\partial_t \mathrm{H}_\infty = \Delta \mathrm{H}_\infty + \mathrm{H}_\infty |\mathrm{A}_\infty|^2$, we actually have $\mathrm{H}_\infty(p, t) > 0$ for every point in space and time, otherwise $\mathrm{H}_\infty$ is identically zero, hence also $\mathrm{A}_\infty$ is zero which is in contradiction with the fact that the limit flow is nonflat.

Fixing any $\eta > 0$ and a pair $(p, s)$ with $p \in M_s^\infty$, if $Q_k \to +\infty$ is the rescaling factor for the flow $\varphi_k$ and $q_k \in M$ is such that $p_k = \varphi_k(q_k, s)$ converges to $p$ as $k \to \infty$, we have $\mathrm{H}_k(q_k, s) = \mathrm{H}(q_k, s/Q_k + t_k)/Q^k \to \mathrm{H}_\infty(p, t) > 0$, hence $\mathrm{H}(q_k, s/Q_k + t_k) \to +\infty$. Now, since by Corollary 4.2.2 there exists a constant $C > 0$ such that $\lambda^{\min} \geq -\eta\mathrm{H} - C$ for the original flow $\varphi$ and $\mathrm{H} > \varepsilon$ at least for every $t > \delta > 0$, we have $\lambda^{\min}/\mathrm{H} \geq -\eta - C/\mathrm{H}$ everywhere. When we rescale the hypersurfaces we get

$$\frac{\lambda_k^{\min}(q_k, s)}{\mathrm{H}_k(q_k, s)} = \frac{\lambda^{\min}(q_k, s/Q_k + t_k)}{\mathrm{H}(q_k, s/Q_k + t_k)} \geq -\eta - \frac{C}{\mathrm{H}(q_k, s/Q_k + t_k)}$$

and sending $k \to \infty$ we conclude $\lambda_\infty^{\min}(p, s)/\mathrm{H}_\infty(p, s) \geq -\eta$.

Since $\eta > 0$ was arbitrary and this argument holds for every pair $(p, s)$ with $p \in M_s^\infty$, the second fundamental form is nonnegative definite on the whole limit flow, hence all the hypersurfaces are convex. $\square$

*Remark* 4.2.4. Instead of using Corollary 4.2.2, one can apply the same argument directly to the estimates of Theorem 4.2.1 obtaining that all the elementary symmetric polynomials of the eigenvalues of the second fundamental form are nonnegative at every point in space and time for the limit flow. By relations (2.5.1) the conclusion follows.

*Remark* 4.2.5. This result also holds if Hamilton's procedure is applied type I singularities (see Exercise 4.1.5).

*Remark* 4.2.6. This proposition (in a slightly stronger form) has also been obtained by White [122] by completely different techniques. His approach also works for the subsequent singularities of "weak" mean curvature flows which continue after the first singular time.

The hypersurfaces of the limit flow are convex, but in general not strictly convex. However, if they are not strictly convex, then they necessarily split as the product of a flat factor with a strictly convex one, as shown by the following result.

**Proposition 4.2.7 (Theorem 4.1 in [73]).** *If any of the convex hypersurfaces of the limit flow $M_s^\infty$ is not strictly convex, then (up to a rigid motion) $M_s^\infty = N_s^m \times \mathbb{R}^{n-m}$, where $1 \leq m \leq n$ and $N_s^m$ is a family of strictly convex, $m$-dimensional, complete hypersurfaces moving by mean curvature in $\mathbb{R}^{m+1}$.*

*Proof.* The proof is based on Hamilton's strong maximum principle for tensors in [56, Section 8] (see Appendix C, Theorem C.1.3), which holds also if the manifold is not compact (as it is in our case).

If $m(s) \in \mathbb{N}$ is the minimal rank of $\mathrm{A}_\infty$ on $M_s^\infty$, arguing as in Remark 2.5.6 this integer-valued function is nondecreasing. Letting $m < n$ be its global minimum which is realized at some point of $M_{s_0}^\infty$, it follows that $m(s) = m$ for every $s \leq s_0$. Again by the argument in Remark 2.5.6, for every $s \leq s_0$ the hypersurface $M_s^\infty$ must contain an $(n-m)$-dimensional affine subspace of $\mathbb{R}^{n+1}$ which is invariant by parallel transport and time. Clearly, such a subspace is the same for all $s \leq s_0$. Thus, the limit flow for $s \in (-\infty, s_0]$ splits as a product of an $(n-m)$-dimensional flat part and a family of strictly convex $m$-dimensional hypersurfaces $N_s^m \subset \mathbb{R}^{m+1}$ evolving by mean curvature.

By uniqueness of the flow as $\mathrm{A}_\infty$ is bounded (see the discussion in Remark 1.5.4), this must hold also for every $s > s_0$. $\square$

**Exercise 4.2.8.** For a type I singularity of the mean curvature flow of a mean convex, *embedded* initial hypersurface, Hamilton's procedure (see Exercise 4.1.5) gives a flow $M_s^\infty$ which is of the form $\mathbb{S}_s^m \times \mathbb{R}^{n-m}$, for some $1 \leq m \leq n$ where $\mathbb{S}_s^m$ is an $m$-dimensional shrinking sphere.

In the case of the evolution of mean convex hypersurfaces in a time interval $[0, T)$, by Proposition 2.4.2 and Corollary 2.4.3, the mean curvature $\mathrm{H}$ and $|\mathrm{A}|$ are comparable quantities, that is, there exists a constant $\alpha$, independent of time such that $\alpha|\mathrm{A}| \leq \mathrm{H} \leq \sqrt{n}|\mathrm{A}|$ for $t \in [\delta, T)$. This implies that we can modify Hamilton's blow-up procedure, substituting $\mathrm{H}^2$ in place of $|\mathrm{A}|^2$ in equation (4.1.1), with the same estimates on the second fundamental form and its covariant derivatives.

We then still get an eternal smooth limit flow, complete with bounded curvature and its covariant derivatives, with the only difference that this time it is the mean curvature H which gets a global maximum equal to one at time zero. This will be crucial to continue the analysis in the next sections.

Analogously, it is easy to see that the conclusions of Propositions 4.2.3 and 4.2.7 are not affected by this modification so also in this case the limit flow consists of convex hypersurfaces.

We call this limit flow Hamilton's *modified* blow-up limit.

*Remark* 4.2.9. Notice that for curves in $\mathbb{R}^2$ the two procedures coincide as $|\mathrm{A}| = |\mathrm{H}| = |k|$, where $k$ is the usual curvature of a curve in the plane.

As the argument leading to Proposition 4.2.3 does not work in the one-dimensional case of curves, we deal with this latter separately in the next section.

## 4.3 The Special Case of Curves

Again, the case of a closed curve in $\mathbb{R}^2$ is special.

We suppose to deal with a generic initial closed curve, smoothly immersed in the plane $\mathbb{R}^2$ and moving by mean curvature $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ where at time $T$ we have a type II singularity. Setting $\xi$ and $k$ to be respectively the arclength and the curvature of $\gamma_t$, we have the evolution equation $\partial_t k = k_{\xi\xi} + k^3$, then we define the function $z(t) = \#\{p \in \gamma_t \,|\, k(p) = 0\}$ "counting" the number of points on $\gamma_t$ such that $k = 0$.

We need the following result of Angenent in [16, Proposition 1.2] and [15, Section 2] (see [13] for the proof).

**Proposition 4.3.1.** *If we have a mean curvature flow of a* (*possibly unbounded*) *curve in $\mathbb{R}^2$ which is not a line, in an open interval of time, at every fixed time the points where $k$ is zero are isolated in space. In particular, this implies that for a closed curve, the function $z$ is finite at every time.*

*The function $z$ is nonincreasing during the flow, hence if at some time it is finite, it remains finite.*

*Finally, if at some point $p \in \gamma_t$ we have $k(p) = 0$ and $k_\xi(p) = 0$, then the zero point $p$ for $k$ immediately vanishes. To be precise, this means that there exists a small space interval $I$ around $p$ and a small $r > t$ such that $k$ is never zero in $I \times (t, r)$.*

We only mention that the proof is based on application of the maximum principle to the above evolution equation for the curvature.

By this proposition, in our case we can define $\mathcal{I}_t$ to be the finite family of open intervals on $\gamma_t$ where $k \neq 0$ and the following computation is justified,

$$\begin{aligned}
\frac{d}{dt}\int_{\gamma_t} |k|\,d\xi &= \sum_{I\in\mathcal{I}_t}\int_I \Big[(\mathrm{sign}k)(k_{\xi\xi}+k^3) - |k|^3\Big]\,d\xi \\
&= \sum_{I\in\mathcal{I}_t}\int_I (\mathrm{sign}k)k_{\xi\xi}\,d\xi \\
&= -\,2 \sum_{\{p\in\gamma_t \,|\, k(p)=0\}} |k_\xi(p)|\,.
\end{aligned}$$

Hence, the integral $\int_{\gamma_t}|k|\,d\xi$, which is positive and finite at every time by compactness, is not increasing during the flow so it converges to some value $L \geq 0$ as $t \to T$, moreover it is scaling invariant.
Then we have, for every $t_1 < t_2$,

$$\int_{\gamma_{t_1}} |k|\,d\xi - \int_{\gamma_{t_2}} |k|\,d\xi = 2\int_{t_1}^{t_2} \sum_{\{p\in\gamma_t \,|\, k(p)=0\}} |k_\xi(p)|\,dt\,.$$

If now we apply Hamilton's procedure, calling $\gamma_s^n$ the rescaled curves at step $n$ with curvatures $k_n$ and denoting by $K_n \to +\infty$ the rescaling factor, we have for every interval $(a,b) \subset \mathbb{R}$,

$$\begin{aligned}
2\int_a^b \sum_{\{p\in\gamma_s^n \,|\, k_n(p)=0\}} |\partial_\xi k_n|\,ds &= \int_{\gamma_a^n} |k_n|\,d\xi - \int_{\gamma_b^n} |k_n|\,d\xi && (4.3.1)\\
&= \int_{\gamma_{\frac{a}{K_n}+t_n}} |k|\,d\xi - \int_{\gamma_{\frac{b}{K_n}+t_n}} |k|\,d\xi\,,
\end{aligned}$$

since $\int_{\gamma_t}|k|\,d\xi$ is scaling invariant and where, by simplicity, we used $\xi$ also for the arclength of the rescaled curves.

It is easy to see that the integral $\int_a^b \sum_{\{p\in\gamma_s \,|\, k(p)=0\}} |k_\xi|\,ds$ is lower semicontinuous under the smooth local convergence of curves, hence

$$\begin{aligned}
\int_a^b \sum_{\{p\in\gamma_s^\infty \,|\, k_\infty(p)=0\}} |\partial_\xi k_\infty|\,ds &\leq \lim_{n\to\infty}\int_a^b \sum_{\{p\in\gamma_s^n \,|\, k_n(p)=0\}} |\partial_\xi k_n|\,ds \\
&= \lim_{n\to\infty}\Big(\int_{\gamma_{\frac{a}{K_n}+t_n}} |k|\,d\xi - \int_{\gamma_{\frac{b}{K_n}+t_n}} |k|\,d\xi\Big) \\
&= 0
\end{aligned}$$

for the limit flow $\gamma_s^\infty$, as both $\frac{a}{K_n}+t_n$ and $\frac{b}{K_n}+t_n$ converge to $T$, hence both integrals in equation (4.3.1) converge to $L$. As $a$ and $b$ were arbitrary, we conclude for almost every $s \in \mathbb{R}$,

$$\sum_{\{p \in \gamma_s^\infty \,|\, k_\infty(p)=0\}} |\partial_\xi k_\infty(p)| = 0\,,$$

that is, $\partial_\xi k_\infty$ is zero at every point in space and time where $k_\infty$ is zero.
Again by means of Proposition 4.3.1, fixing $s \in \mathbb{R}$ and choosing any small $r > s$, the zero points of the curvature vanish for the curve $\gamma_r^\infty$, hence $k_\infty > 0$ on $\gamma_r^\infty$ for every $r > s$, as it is a condition which is preserved under the flow. Since we can draw this conclusion for almost every $s \in \mathbb{R}$, at every time the flow $\gamma_s^\infty$ consists of curves such that $k_\infty$ is never zero.
Hence, we have the following one-dimensional analogue of Proposition 4.2.3.

**Proposition 4.3.2.** *The limit flow $\gamma_s^\infty$ obtained by Hamilton's procedure at a type II singularity of the evolution by curvature of any initial closed curve, consists of curves such that $k_\infty$ is never zero, in particular if the initial curve was embedded all such curves are strictly convex.*

*Remark* 4.3.3. We underline that we did not assume that the initial curve was embedded. The above conclusion holds for the flow of any immersed closed curve in the plane (like the results of the previous section holding for general immersed-only hypersurfaces).

## 4.4 Hamilton's Harnack Estimate for Mean Curvature Flow

We have seen in the previous two sections that if a closed curve or a compact hypersurface with $\mathrm{H} \geq 0$ develops a type II singularity, then the limit of the rescaled flows by the "modified" Hamilton's procedure is an eternal mean curvature flow of convex, complete, hypersurfaces such that H takes its maximum in space and time at some point. We want now to see that this implies that such limit flow is translating, which is obtained by means of the following two deep results of Hamilton in [62].

**Theorem 4.4.1 (Harnack Estimate for Mean Curvature Flow).** *Let $\varphi : M \times (T_0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a complete, convex hypersurface with bounded second fundamental form at every time.*
*Let $X$ be a time-dependent, smooth tangent vector field on $M$. Then the following inequality holds,*

$$\frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t - T_0)} + 2\langle \nabla \mathrm{H} \,|\, X \rangle + h_{ij} X^i X^j \geq 0$$

*for every $t \in (T_0, T)$.*

**Theorem 4.4.2.** *Let $\varphi : M \times (-\infty, T) \to \mathbb{R}^{n+1}$ be an ancient mean curvature flow of a complete, strictly convex hypersurface with bounded second fundamental form at every time and such that* H *takes its maximum in space and time. Then, $\varphi$ is a translating flow.*

The proofs of these two theorems involve some smart and heavy computations with a strong use of the maximum principle; we show the complete proof only in the one-dimensional and compact case, referring the reader to the original paper [62] (see also [57]).

*Proof of Theorem* 4.4.1 – *One-dimensional Compact Case.* As the evolving curves are compact, the curvature $k$ and all its derivatives are bounded in $(C, T - \varepsilon)$, for every $\varepsilon > 0$ and $C \in (T_0, T - \varepsilon)$. Moreover, by Proposition 2.4.1, in the same interval, $k > k_0 > 0$ for some positive constant $k_0$.

Since any tangent vector field $X$ can be written as $X = \lambda\tau$ for some function $\lambda : \mathbb{S}^1 \times (T_0, T) \to \mathbb{R}$, we define the Hamilton quadratic

$$Z(\lambda) = \partial_t k + \frac{k}{2(t-C)} + 2\lambda k_s + k\lambda^2 = k_{ss} + k^3 + \frac{k}{2(t-C)} + 2\lambda k_s + k\lambda^2$$

which is clearly bounded from below by

$$Z = k_{ss} + k^3 - k_s^2/k + \frac{k}{2(t-C)} \,.$$

We also define

$$W = k_{ss} + k^3 - k_s^2/k$$

and we start computing the evolution equation for this latter quantity by means of the evolution equations in Remark 2.3.2,

$$\begin{aligned}
(\partial_t - \partial_{ss})W &= \partial_t k_{ss} - \frac{2k_s\partial_t k_s}{k} + \frac{k_s^2 k_t}{k^2} + 3k^2 k_t - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} \\
&\quad - \frac{5k_s^2 k_{ss}}{k^2} + \frac{2k_s^4}{k^3} - 6kk_s^2 - 3k^2 k_{ss} \\
&= \partial_s\partial_t k_s + k^2 k_{ss} - \frac{2k_s\partial_s k_t}{k} - 2kk_s^2 + \frac{k_s^2 k_{ss}}{k^2} + kk_s^2 + 3k^2 k_{ss} + 3k^5 \\
&\quad - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} - \frac{5k_s^2 k_{ss}}{k^2} + \frac{2k_s^4}{k^3} - 6kk_s^2 - 3k^2 k_{ss} \\
&= \partial_{ss}(k_{ss} + k^3) + 2k^2 k_{ss} - 5kk_s^2 - \frac{2k_s\partial_s(k_{ss} + k^3)}{k} \\
&\quad + \frac{k_s^2 k_{ss}}{k^2} + 3k^5 - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} - \frac{5k_s^2 k_{ss}}{k^2} + \frac{2k_s^4}{k^3}
\end{aligned}$$

$$
\begin{aligned}
&= k_{ssss} + 5k^2 k_{ss} - 5kk_s^2 - \frac{2k_s k_{sss}}{k} \\
&\quad - \frac{4k_s^2 k_{ss}}{k^2} + 3k^5 - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} + \frac{2k_s^4}{k^3} \\
&= -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} + 5k^2 k_{ss} + \frac{2k_{ss}^2}{k} - \frac{4k_s^2 k_{ss}}{k^2}\,.
\end{aligned}
$$

As $k_{ss} = (W + k_s^2/k - k^3)$, substituting we get

$$
\begin{aligned}
(\partial_t - \partial_{ss})W = &-5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} \\
&+ 5k^2(W + k_s^2/k - k^3) \\
&+ \frac{2(W + k_s^2/k - k^3)^2}{k} - \frac{4k_s^2(W + k_s^2/k - k^3)}{k^2} \\
= &-5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} \\
&+ 5k^2 W + 5kk_s^2 - 5k^5 \\
&+ \frac{2W^2}{k} + \frac{2k_s^4}{k^3} + 2k^5 + \frac{4Wk_s^2}{k^2} - 4Wk^2 - 4kk_s^2 \\
&- \frac{4k_s^2(W + k_s^2/k - k^3)}{k^2} \\
= &\frac{2W^2}{k} + Wk^2\,.
\end{aligned}
\tag{4.4.1}
$$

We notice that, since $k > k_0 > 0$, by the maximum principle if $W$ is positive at some time it remains positive.
As $Z = W + k/(2(t - C))$, we then get

$$
\begin{aligned}
(\partial_t - \partial_{ss})Z &= (\partial_t - \partial_{ss})W + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
&= \frac{2W^2}{k} + Wk^2 + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
&= \frac{2(Z - k/(2(t-C)))^2 + k^3(Z - k/(2(t-C)))}{k} + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
&= \frac{2Z^2 + k^2/(2(t-C)^2) - 2Zk/(t-C)}{k} + \frac{k^3 Z - k^4/(2(t-C))}{k} \\
&\quad + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2} \\
&= \frac{2Z^2}{k} - \frac{2Z}{t-C} + k^2 Z\,.
\end{aligned}
$$

As $k > k_0 > 0$ the term $k/(2(t - C))$ diverges as $t \to C^+$ and $W$ is bounded from below, then we have that $Z$ goes uniformly to $+\infty$ as $t \to C^+$. Hence, $Z$ is positive

in $\mathbb{S}^1 \times (C, C+\delta)$ for some $\delta > 0$ and by the maximum principle it cannot be zero on $\gamma_t$ for every $t \in (C, T-\varepsilon)$.
As $Z(\lambda) \geq Z > 0$ for every function $\lambda : M \times (T_0, T) \to \mathbb{R}$, sending $\varepsilon \to 0$ and $C \to T_0$ we have the thesis of the theorem. □

*Remark* 4.4.3. When the curves $\gamma_t$ are not compact there are two nontrivial technical points to take care of: the possible nonexistence of the minimum in space of $Z(\,\cdot\,, t)$ and the fact that it is not granted that $\lim_{t\to C^+} \inf_{\gamma_t} Z(\,\cdot\,, t) = +\infty$, as $k$ could go to zero at infinity (possibly, a value $k_0 > 0$ such that $k > k_0$ uniformly does not exist). This requires a perturbation of $Z$ in space with a function growing enough at infinity and the addition to $Z$ of another function assuring that the resulting term uniformly diverges as $t \to C^+$ (see [57, 62] for the details).

*Remark* 4.4.4. The higher complexity of the proof in dimension larger than one is essentially due to the fact that the minimum of the quadratic

$$Z(X) = \frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-C)} + 2\langle \nabla \mathrm{H} \,|\, X \rangle + h_{ij} X^i X^j \,,$$

which is given by

$$Z = \frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-C)} - (\mathrm{A}^{-1})^{pq} \nabla_p \mathrm{H} \nabla_q \mathrm{H}\,,$$

is clearly more difficult to deal with than in the one-dimensional case (here $(\mathrm{A}^{-1})^{pq}$ denotes the *inverse matrix* of the second fundamental form $h_{ij}$, that is

$$(\mathrm{A}^{-1})^{pq} h_{qr} = \delta^p_r).$$

Anyway, after a quite long computation one can see that

$$\frac{\partial Z}{\partial t} - \Delta Z = 2 g^{ij} (\mathrm{A}^{-1})^{kl} J_{ik} J_{jl} + \Big(|\mathrm{A}|^2 - \frac{2}{t-C}\Big) Z \geq \Big(|\mathrm{A}|^2 - \frac{2}{t-C}\Big) Z$$

where

$$J_{ik} = \nabla^2_{ik} \mathrm{H} + \mathrm{H} h^2_{ik} - (\mathrm{A}^{-1})^{pq} \nabla_p \mathrm{H} \nabla_q h_{ik} + \frac{h_{ik}}{2(t-C)}\,,$$

see [27, Chapter 15].
Actually, another possibility is to keep the vector field $X$ generic and to compute the evolution equation for $Z(X)$, as in the original proof of Hamilton.

*Proof of Theorem* 4.4.2 – *One-dimensional Compact Case.* Suppose that we have an ancient curvature flow $\gamma_t$ of closed curves in the plane with $k > 0$. By Theorem 4.4.1 we have

$$Z = \partial_t k - k_s^2/k + k/(t - T_0) \geq 0$$

at every point and for every $t, T_0 \in \mathbb{R}$ with $T_0 < t < T$. Sending $T_0 \to -\infty$ we get

$$W = \partial_t k - k_s^2/k \geq 0\,.$$

Since we computed in equation (4.4.1) that

$$(\partial_t - \partial_{ss})W = \frac{2W^2}{k} + Wk^2\,,$$

if $W$ is zero at some point in space and time, it must be zero everywhere by the maximum principle. By hypothesis $k$ takes a maximum at some point in space and time, hence at such a point $k_t = k_s = 0$ which implies $W = 0$.
Thus, $k_t = k_s^2/k$ for all the curves of flow, or equivalently $k_{ss} + k^3 - k_s^2/k = 0$.
If we set $v = -(k_s/k)\tau + k\nu$ as a vector field in $\mathbb{R}^2$ along $\gamma_t$, obviously $\langle v\,|\,\nu\rangle = k$, then

$$\begin{aligned}
\partial_s v =& -(k_{ss}/k - k_s^2/k^2)\tau - (k_s/k)k\nu + k_s\nu - k^2\tau\\
=& -(-k^2 + k_s^2/k^2 - k_s^2/k^2)\tau - k^2\tau = 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_t v =& (-k_{ts}/k + k_s k_t/k^2 - kk_s)\tau + (-k_s^2/k + k_t)\nu\\
=& (-k_{st}/k - kk_s + k_s^3/k^3 - kk_s)\tau\\
=& (-[\partial_s(k_s^2/k)]/k + k_s^3/k^3 - 2kk_s)\tau\\
=& (-2k_s k_{ss}/k^2 + 2k_s^2/k^3 - 2kk_s)\tau\\
=& -2\frac{k_s}{k}(k_{ss} - k_s^2/k + k^3)\tau = 0\,.
\end{aligned}$$

Hence, as the curves of the limit flow are connected, $v$ is a vector field along $\gamma_t$ constant in space and time.
As $k = \langle v\,|\,\nu\rangle$, we have that $\gamma_t$ moves by translation under the mean curvature flow. $\square$

Then, putting together Propositions 4.2.3, 4.2.7, 4.3.2 and Theorem 4.4.2, we have the following results.

**Theorem 4.4.5.** *The blow-up limit flow obtained by Hamilton's* modified *procedure at a type II singularity of the motion of a initial hypersurface with* $\mathrm{H} \geq 0$ *is a translating mean curvature flow of complete, nonflat, convex hypersurfaces with bounded curvature and its covariant derivatives.*
*If any of the convex hypersurfaces of the limit flow is not strictly convex, then the limit flow splits as the product of an* $m$*-dimensional strictly convex, translating flow as above and* $\mathbb{R}^{n-m}$.

**Theorem 4.4.6.** *The blow-up limit flow obtained by Hamilton's procedure at a type II singularity of the motion of a closed curve in the plane is a translating curvature flow of complete, nonflat curves with bounded curvature and its covariant derivatives. Moreover, for all the curves,* $k > 0$.
*Hence, this flow is given (up to rigid motions) by the* grim reaper (*see Section* 1.4).

*Remark* 4.4.7. For curves in the plane, possibly with self-intersections, such that the initial curvature is never zero, this result was obtained via a different method by Angenent [15] (see also [3]), studying directly the parabolic equation satisfied by the curvature function.

In [122], White was able to exclude the possibility of getting as a blow-up limit the product of a grim reaper with $\mathbb{R}^{n-1}$, when $n \geq 2$.

In dimension two, by this result of White and the analysis of Wang [119], the only possible blow-up limit flow is given (up to a rigid motion) by the unique rotationally symmetric, translating hypersurface which is the graph of an entire, strictly convex function described by the ODE (1.4.1), in Section 1.4.

In general, without assuming the condition $\mathrm{H} \geq 0$, one could conjecture that blow-up limits like the minimal catenoid surface $M$ in $\mathbb{R}^3$ given by

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \mid \cosh|y| = |x|\big\}$$

cannot appear. See White [122], Ecker [36] and the recent paper by Sheng and Wang [103] for more details.

## 4.5 Embedded Closed Curves in the Plane

In the special case of the evolution of an embedded closed curve in the plane, it is possible to exclude at all type II singularities. This, together with the case of convex, compact, hypersurfaces (as we have seen in the proof of Theorem 3.4.10) are the only known cases in which this can be done in general.

By Theorem 4.4.6 and embeddedness, any blow-up limit must be a unit multiplicity grim reaper. We apply now a very geometric argument by Huisken in [70] in order to exclude also such a possibility (see also [63] for another similar quantity).

Given the smooth flow $\gamma_t$ of an initial embedded closed curve in some interval $[0, T)$, we know that the curve stays embedded during the flow so we can see every $\gamma_t$ as a subset of $\mathbb{R}^2$. At every time $t \in [0, T)$, for every pair of points $p$ and $q$ in $\gamma_t$ we define $d_t(p, q)$ to be the *geodesic* distance in $\gamma_t$ of $p$ and $q$, $|p - q|$ the standard distance in $\mathbb{R}^2$ and $L_t$ the length of $\gamma_t$.

We consider the function $\Phi_t : \gamma_t \times \gamma_t \to \mathbb{R}$ defined as

$$\Phi_t(p, q) = \begin{cases} \frac{\pi|p-q|}{L_t} \Big/ \sin \frac{\pi d_t(p,q)}{L_t} & \text{if } p \neq q, \\ 1 & \text{if } p = q\,, \end{cases}$$

which is a perturbation of the quotient between the extrinsic and the intrinsic distance of a pair of points on $\gamma_t$.

Since $\gamma_t$ is smooth and embedded at every time, the function $\Phi_t$ is well defined and positive. Moreover, it is easy to check that, even if $d_t$ is not $C^1$ at the pairs of points such that $d_t(p,q) = L_t/2$, the function $\Phi_t$ is $C^2$ in the open set $\{p \neq q\} \subset \gamma_t \times \gamma_t$ and continuous on $\gamma_t \times \gamma_t$.

By compactness, the following minimum exists,

$$E(t) = \min_{p,q \in \gamma_t} \Phi_t(p,q)\,.$$

We call this quantity *Huisken's embeddedness ratio.*

Since the evolution is smooth it is easy to see that the function $E : [0,T) \to \mathbb{R}$ is continuous.

*Remark* 4.5.1. The quantity $E$ can be defined also for nonembedded closed curves, but in such case $E = 0$; indeed its positivity is equivalent to embeddedness.

**Lemma 4.5.2 (Huisken [70]).** *The function $E(t)$ is monotone increasing in every time interval where $E(t) < 1$.*

*Proof.* We start differentiating in time $\Phi_t(p,q)$,

$$\begin{aligned}
&\frac{d}{dt}\Phi_t(p,q)\\
&\quad= \frac{\pi}{L_t}\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&\qquad+\left(\frac{\pi|p-q|}{L_t^2}\int_{\gamma_t}k^2\,ds\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&\qquad-\frac{\pi^2|p-q|}{L_t^2}\cos\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\Big/\sin^2\frac{\pi d_t(p,q)}{L_t}\\
&\quad=\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\int_{\gamma_t}k^2\,ds\right.\\
&\qquad\left.-\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\right]\Phi_t(p,q)\\
&\quad=\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\left(1-\frac{\pi d_t(p,q)}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\right)\int_{\gamma_t}k^2\,ds\right.\\
&\qquad\left.+\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\int_q^p k^2\,ds\right]\Phi_t(p,q)
\end{aligned}$$

where $s$ is the arclength and $k$ the curvature of $\gamma_t$. It is easy to see that since the function $E$ is the minimum of a family of uniformly locally Lipschitz functions, it is also locally Lipschitz, hence differentiable almost everywhere. Then, to prove the statement it is enough to show that $\frac{dE(t)}{dt} > 0$ for every time $t$ such that this derivative exists. We will do that as usual, by Hamilton's trick (Lemma 2.1.3).

Let $(p,q)$ be a minimizing pair at a differentiability time $t$ and suppose that $E(t) < 1$. By the very definition of $\Phi_t$, it must be $p \neq q$.

We set $\alpha = \pi d_t(p,q)/L_t$ and notice that $\alpha \cot \alpha < 1$ as $\alpha \in (0, \pi/2]$. Moreover, $\int_{\gamma_t} k^2\, ds \geq \left(\int_{\gamma_t} k\, ds\right)^2 / L_t \geq 4\pi^2/L_t$. Then, we have

$$\frac{d}{dt}E(t) \geq \left[\frac{\langle p-q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha \int_q^p k^2\, ds\right] E(t)$$

that is,

$$\frac{d}{dt}\log E(t) \geq \frac{\langle p-q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha \int_q^p k^2\, ds\,, \tag{4.5.1}$$

for any minimizing pair $(p, q)$.
Assume that the curve is parametrized counterclockwise by arclength and that $p$ and $q$ are as shown in Figure 2.

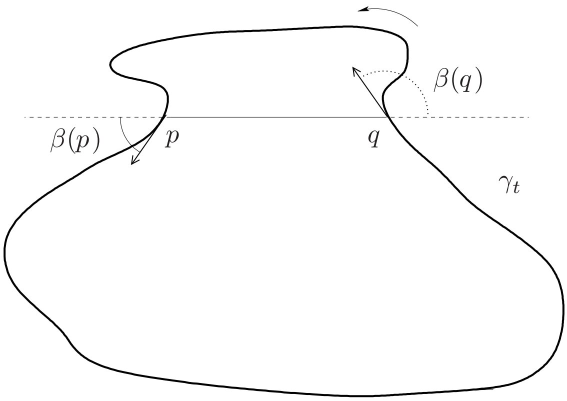


Figure 2.

We set $p(s) = \gamma_t(s_1 + s)$ with $p = \gamma_t(s_1)$, then, by minimality we have

$$0 = \frac{d}{ds}\Phi_t(p(s), q)\bigg|_{s=0} = \frac{\pi}{L_t}\frac{\langle p-q \,|\, \tau(p)\rangle}{|p-q| \sin\frac{\pi d_t(p,q)}{L_t}} - \frac{\pi|p-q|}{L_t \sin^2\frac{\pi d_t(p,q)}{L_t}} \cdot \frac{\pi\cos\frac{\pi d_t(p,q)}{L_t}}{L_t}$$

where we have denoted by $\tau(p)$ the oriented unit tangent vector to $\gamma_t$ at $p$.
By this equality we get

$$\cos\beta(p) = \frac{\langle p-q \,|\, \tau(p)\rangle}{|p-q|} = \frac{\pi|p-q|}{L_t \sin\frac{\pi d_t(p,q)}{L_t}} \cos\frac{\pi d_t(p,q)}{L_t} = E(t)\cos\alpha$$

where $\beta(p) \in [0, \pi/2]$ is the angle between the vectors $p - q$ and $\tau(p)$.

Repeating this argument for the point $q$ we get

$$\cos\beta(q) = -E(t)\cos\alpha$$

where, as before, $\beta(q)$ is the angle between $q-p$ and $\tau(q)$, see Figure 2. Clearly, it follows that $\beta(p)+\beta(q)=\pi$.
Notice that if one of the intersections of the segment $[p,q]$ with the curve is tangential, we would have $E(t)\cos\alpha = 1$ which is impossible as we assumed that $E(t)<1$. Moreover, by the relation $\cos\beta(p) = E(t)\cos\alpha < \cos\alpha$ it follows that $\beta(p)>\alpha$.

We look now at the second variation of $\Phi_t$ at the same minimizing pair of points $(p,q)$. With the same notation, if $p=\gamma_t(s_1)$ and $q=\gamma_t(s_2)$ we set $p(s)=\gamma_t(s_1+s)$ and $q(s)=\gamma_t(s_2-s)$. After a straightforward computation, one gets

$$\begin{aligned}
0 \leq\ & \frac{d^2}{ds^2}\Phi_t(p(s),q(s))\bigg|_{s=0}\\
&= \frac{\pi}{L_t}\left(\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|} + \frac{4\pi^2|p-q|}{L_t^2}\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&= \left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}\right]E(t)\,.
\end{aligned}$$

Hence, getting back to inequality (4.5.1) we have

$$\begin{aligned}
\frac{d}{dt}\log E(t) \geq & \frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\\
\geq & -\frac{4\pi^2}{L_t^2}\alpha\cot\alpha + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\\
= & \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right),
\end{aligned}$$

so it remains to show that this last expression is positive. As

$$\int_p^q k^2\,ds \geq \left(\int_p^q k\,ds\right)^2 \Big/ d_t(p,q)$$

and noticing that $\int_p^q k\,ds$ is the angle between the tangent vectors $\tau(p)$ and $\tau(q)$ we have $\left(\int_p^q k\,ds\right)^2 = 4\beta(p)^2 < 4\alpha^2$, as we concluded before.
Thus,

$$\begin{aligned}
\frac{d}{dt}\log E(t) \geq & \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right)\\
> & \frac{\pi\cot\alpha}{L_t}\left(\frac{4\alpha^2}{d_t(p,q)} - \frac{4\pi}{L_t}\alpha\right) = 0
\end{aligned}$$

recalling that $\alpha = \pi d_t(p,q)/L_t$. $\square$

*Remark* 4.5.3. By its definition and Lemma 4.5.2, the function $E$ is always nondecreasing. Actually, to be more precise, by means of a simple geometric argument it can be proved that if $E(t) = 1$ the curve $\gamma_t$ must be a circle. Hence, in any other case $E$ is strictly increasing in time.

An immediate consequence is that for every initial embedded, closed curve in $\mathbb{R}^2$, there exists a positive constant $C$, depending on the initial curve, such that on all $[0, T)$ we have $E(t) \geq C$. The same conclusion holds for any rescaling of such curves as the function $E$ is scaling invariant by construction.

*Remark* 4.5.4. This lemma also provides an alternative proof of the fact that an initial embedded, closed curve stays embedded. Indeed, it cannot develop a self-intersection during its curvature flow, otherwise $E$ would go to zero.

We can then exclude type II singularities in the curvature flow of embedded closed curves.
Any blow-up limit flow $\gamma^\infty$ is given (up to rigid motions) by a grim reaper, that is, the translating graph $\Gamma$ of the function $y = -\log\cos x$ in the interval $(-\pi/2, \pi/2)$. Assuming that $\gamma^\infty_0 = \Gamma$, we consider the following four points: $p_1 = (-x_1, -\log\cos x_1)$, $q_1 = (x_1, -\log\cos x_1)$, $p_2 = (-x_2, -\log\cos x_2)$ and $q_2 = (x_2, -\log\cos x_2)$ belonging to $\Gamma$, for $0 < x_1 < x_2 < \pi/2$ such that $-\log\cos x_2 > \pi/2 - 3\log\cos x_1$.

As the rescaled curves $\gamma^k_0$ converge locally in $C^1$ to $\Gamma$, for any $\varepsilon > 0$ such that $x_2 + \varepsilon < \pi/2$ and $k$ is large enough, the curve $\gamma^k_0$ will be $C^1$-close to $\Gamma$ in the open rectangle $R_\varepsilon = (-x_2 - \varepsilon, x_2 + \varepsilon) \times (-\varepsilon, -\log\cos x_2 + \varepsilon)$, hence there will be a pair of points $(p, q) \in \gamma^k_0$ arbitrarily close to $(p_1, q_1)$ and another pair $(\widetilde{p}, \widetilde{q}) \in \gamma^k_0$ arbitrarily close to $(p_2, q_2)$. As $k \to \infty$, the geodesic distance $d_{\gamma^k_0}(p, q)$ on the closed curve $\gamma^k_0$ between $p$ and $q$ is definitely given by the length of the part of the curve which is close to the vertex of $\Gamma$. Indeed, this latter is smaller than $\pi - 2\log\cos x_1$, when $k$ is large enough, instead the *other part* of the curve has a length which is at least the sum of the Euclidean distances $|\widetilde{p} - p| + |\widetilde{q} - q|$ which is definitely larger than $2(\log\cos x_1 - \log\cos x_2)$ and this last quantity is larger than $\pi - 4\log\cos x_1$, by construction.

Hence, when $k$ is large enough, the Huisken's embeddedness ratio for the rescaled curve $\gamma^k_0$ is not larger than

$$\begin{aligned}
\frac{\pi|p-q|}{L}\Big/\sin\frac{\pi d_{\gamma^k_0}(p,q)}{L} &\leq \frac{\pi(\pi+2\varepsilon)}{L}\Big/\sin\frac{\pi d_{\gamma^k_0}(p,q)}{L}\\
&\leq \frac{\pi(\pi+2\varepsilon)}{L}\Big/\frac{2d_{\gamma^k_0}(p,q)}{L}\\
&= \frac{\pi(\pi+2\varepsilon)}{2d_{\gamma^k_0}(p,q)} \leq \frac{\pi^2}{d_{\gamma^k_0}(p,q)}\,,
\end{aligned}$$

where $L$ is the total length of the curve $\gamma^k_0$ and we used the inequality $\sin x \geq 2x/\pi$ holding for every $x \in [0, \pi/2]$.

Finally, again by the $C^1$-convergence of $\gamma_0^k$ to $\Gamma$ in $R_\varepsilon$, we can also assume that $d_{\gamma_0^k}(p,q)$ is larger than $-\log\cos x_1$.

Now we consider a sequence of pairs $x_1^i < x_2^i$ as above such that $x_1^i \to \pi/2$, then we have a sequence of rescaled curves $\gamma_0^{k_i}$ such that the associated Huisken's embeddedness ratio tends to zero, as $d_{\gamma_0^{k_i}}(p,q) \to +\infty$ when $i \to \infty$.

This is in contradiction with the fact that the function $E$ is scaling invariant and uniformly bounded from below by some positive constant $C$ for all the curves of the flow.

As this argument does not change if we apply to $\Gamma$ any rigid motion, in presence of a type II singularity in the embedded case, we would have a contradiction with the conclusion of Theorem 4.4.6.

**Theorem 4.5.5.** *Type II singularities cannot develop during the curvature flow of an embedded, closed curve in* $\mathbb{R}^2$.

Collecting together Theorem 3.5.1 about type I singularities and this last proposition, we obtain Theorem 3.4.9 by Gage and Hamilton and the following theorem due to Grayson [51], whose original proof is more geometric and direct, showing that the intervals of negative curvature vanish in finite time before any singularity. We underline that the success of the line of proof we followed is due to the bound from below on Huisken's embeddedness ratio implied by Lemma 4.5.2.

Modifying a little such a quantity, Andrews and Bryan [12] were even able to give a simple and direct proof without passing through the classification of singularities.

**Theorem 4.5.6 (Grayson's Theorem).** *Let $\gamma_t$ be the curvature flow of a closed, embedded, smooth curve in the plane, in the maximal interval of smooth existence* $[0,T)$.

*Then, there exists a time $\tau < T$ such that $\gamma_\tau$ is convex.*

*As a consequence, the result of Gage and Hamilton Theorem* 3.4.9 *applies and subsequently the curve shrinks smoothly to a point as $t \to T$.*

*Remark* 4.5.7. This result, extended by Grayson to curves moving inside general surfaces, allowed him to have a proof of the *three geodesics theorem* on the sphere [53] (first outlined by Lusternik and Schnirelman in [89]).

We add a final remark in this case of embedded closed curves.

Letting $A(t)$ be the area enclosed by $\gamma_t$ which moves by curvature, we have

$$\frac{d}{dt}A(t) = -\int_{\gamma_t} k\,ds = -2\pi\,,$$

hence, as the evolution is smooth till the curve shrinks to a point at time $T > 0$ and clearly $A(t)$ goes to zero, we have $A(0) = 2\pi T$. That is, the maximal time of existence is exactly equal to the initially enclosed area divided by $2\pi$.

### 4.5.1 An Alternative Proof of Grayson's Theorem

Ideas and techniques are related to the unpublished work of Ilmanen [79].

In the special case of embedded curves in the plane, one can avoid the use of Hamilton's Harnack inequality in order to deal with type II singularities. By means of Huisken's monotonicity formula we can produce a homothetic blow-up limit also in the type II case.

As underlined in Remark 3.2.23, White's Theorem 3.2.22 holds in general, without assuming any blow-up rate on the curvature, hence at a singularity time $T > 0$ we have that $\Sigma > 1$ (recall Definition 3.2.3). Moreover, the estimates in Lemma 3.2.7 are also independent of the type I hypothesis.
Then, rescaling the curves around the moving points $x_t$ as in Remark 3.3.9, we have

$$\sigma(0) - \Sigma = \int_{-\frac{1}{2}\log T}^{+\infty} \int_{\gamma_r} e^{-\frac{|y|^2}{2}} \left| \widetilde{k} + \langle y \,|\, \widetilde{\nu} \rangle \right|^2 ds\, dr < +\infty\,.$$

Clearly, since we are not assuming the type I hypothesis the curvatures $\widetilde{k}$ of the rescaled curves $\widetilde{\gamma}_r$ are not bounded, but by this formula it follows that for every family of disjoint intervals $(a_i, b_i) \subset [-\frac{1}{2}\log T, +\infty)$ such that $\sum_{i\in\mathbb{N}}(b_i - a_i) = +\infty$ we can find a sequence $r_i \in (a_i, b_i)$ such that $r_i \nearrow +\infty$,

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}} \left| \widetilde{k} + \langle y \,|\, \widetilde{\nu} \rangle \right|^2 ds = 0$$

and

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}} ds = \lim_{i\to\infty} \sigma(t(r_i)) = \Sigma\,. \tag{4.5.2}$$

By the estimate (3.2.6) on the local length, it follows that the sequence of curves $\widetilde{\gamma}_{r_i}$ has curvatures *locally* equibounded in $L^2$. Hence, we can extract a subsequence which converges in $C^1_{\text{loc}}$ to a limit curve $\widetilde{\gamma}_\infty$. Such a limit curve satisfies $\widetilde{k} + \langle x \,|\, \widetilde{\nu} \rangle = 0$, as the integral $\int_{\widetilde{\gamma}} e^{-\frac{|y|^2}{2}} \left| \widetilde{k} + \langle y \,|\, \widetilde{\nu} \rangle \right|^2 ds$ is lower semicontinuous under $C^1_{\text{loc}}$-convergence. Moreover, by a bootstrap argument $\widetilde{\gamma}_\infty$ is smooth.
By the energy argument in the proof of Proposition 3.4.1 and the length estimate in Lemma 3.2.7, this limit curve is either a line (with possible integer multiplicity) or it is bounded, hence closed and the convergence is actually in $C^1$. As the initial curve was embedded, the Huisken's embeddedness ratio $E$ is uniformly bounded from below on the sequence of rescaled curves, this implies that also $\widetilde{\gamma}_\infty$ is embedded. Indeed, if it has self-intersections or multiplicities the quantity $E$ must approach zero, in the case of a closed limit curve because of the $C^1$-convergence, in the case that the limit curve is a line, by means of the same argument used to exclude the grim reaper in the proof of Theorem 4.5.5.
Hence, by the classification theorem Proposition 3.4.1 we conclude that there are only two possibilities for $\widetilde{\gamma}_\infty$, either a line through the origin of $\mathbb{R}^2$ or the unit circle, both with unit multiplicity.

Since the second point of Lemma 3.2.7 implies that

$$\lim_{i\to\infty}\frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}}\,ds = \frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{\infty}} e^{-\frac{|y|^2}{2}}\,ds\,,$$

and the first limit is equal to $\Sigma > 1$ by equation (4.5.2), we conclude that $\widetilde{\gamma}_\infty$ is the unit circle. Moreover, the curvatures of the converging sequence of curves are equibounded in $L^2$ (not only locally).

Fixing $i \in \mathbb{N}$ and letting $\rho = r - r_i < 1$, as $r = -\frac{1}{2}\log 2(T-t)$, recalling the formulas in Remark 2.3.2 we compute the evolution of the following quantity:

$$\begin{aligned}
&\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\\
&\quad= 2(T-t)\frac{d}{dt}\int_{\gamma_t}\sqrt{2(T-t)}\,k^2\,ds + \int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\\
&\qquad+ 2(T-t)\rho\frac{d}{dt}\int_{\gamma_t}(\sqrt{2(T-t)})^3\,k_s^2\,ds\\
&\quad= -\sqrt{2(T-t)}\int_{\gamma_t}k^2\,ds + (\sqrt{2(T-t)})^3\int_{\gamma_t}(2kk_{ss}+k^4)\,ds\\
&\qquad+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds - 3\big(\sqrt{2(T-t)}\big)^3\rho\int_{\gamma_t}k_s^2\,ds\\
&\qquad+\big(\sqrt{2(T-t)}\big)^5\rho\int_{\gamma_t}(2k_sk_{sss}+7k^2k_s^2)\,ds\\
&\quad=\int_{\widetilde{\gamma}_r}\Big[-\widetilde{k}^2+2\widetilde{k}\widetilde{k}_{ss}+\widetilde{k}^4+\widetilde{k}_s^2-3\rho\widetilde{k}_s^2+2\rho\widetilde{k}_s\widetilde{k}_{sss}+7\rho\widetilde{k}^2\widetilde{k}_s^2\Big]\,ds\,,
\end{aligned}$$

where we used the formula $\partial_t k_s = \partial_s\partial_t k + k^2 k_s = k_{sss} + 4k^2 k_s$.
By integration by parts and Peter–Paul inequality, we have

$$\int_{\widetilde{\gamma}_r}\widetilde{k}^2\widetilde{k}_s^2\,ds = \frac{1}{3}\int_{\widetilde{\gamma}_r}\partial_s(\widetilde{k}^3)\widetilde{k}_s\,ds = -\frac{1}{3}\int_{\widetilde{\gamma}_r}\widetilde{k}^3\widetilde{k}_{ss}\,ds \le \frac{1}{6}\int_{\widetilde{\gamma}_r}\widetilde{k}^6+\widetilde{k}_{ss}^2\,ds$$

and

$$\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds &\le \int_{\widetilde{\gamma}_r}\Big[-\widetilde{k}_s^2+\widetilde{k}^4-\widetilde{k}^2-3\rho\widetilde{k}_s^2-2\rho\widetilde{k}_{ss}^2+7\rho(\widetilde{k}^6+\widetilde{k}_{ss}^2)/6\Big]\,ds\\
&\le \int_{\widetilde{\gamma}_r}(-\widetilde{k}_s^2+\widetilde{k}^4+3\rho\widetilde{k}^6)\,ds\,.
\end{aligned}$$

Now, the following interpolation inequalities for any closed curve in the plane of length $L$ (see Aubin [20, page 93])

$$\|\widetilde{k}\|_{L^4}^4 \le C\|\widetilde{k}_s\|_{L^2}\|\widetilde{k}\|_{L^2}^3 + \frac{C}{L}\|\widetilde{k}\|_{L^2}^4 \quad\text{and}\quad \|\widetilde{k}\|_{L^6}^6 \le C\|\widetilde{k}_s\|_{L^2}^2\|\widetilde{k}\|_{L^2}^4 + \frac{C}{L^2}\|\widetilde{k}\|_{L^2}^6$$

imply, by means of Young inequality,

$$\int_{\widetilde{\gamma}_r} \widetilde{k}^4\, ds \leq \frac{1}{2}\int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\, ds + \frac{1}{2}\Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3 + \Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3 + \frac{C}{L^3(\widetilde{\gamma}_r)}$$

and

$$3\rho\int_{\widetilde{\gamma}_r} \widetilde{k}^6\, ds \leq \Big(\rho\int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\, ds\Big)^3 + 2\Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3 + \frac{C}{L^2(\widetilde{\gamma}_r)}\Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3 .$$

Hence, as we know that $L(\widetilde{\gamma}_r) \geq \int_{\widetilde{\gamma}_r} e^{-\frac{|y|^2}{2}}\, ds \geq \sqrt{2\pi}$ and $\rho < 1$, we conclude that

$$\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + \rho\widetilde{k}_s^2)\, ds \leq & \int_{\widetilde{\gamma}_r} (-\widetilde{k}_s^2 + \widetilde{k}_s^2/2)\, ds + C\Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3 + C\\
& + \Big(\rho\int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\, ds\Big)^3 + C\Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3\\
\leq & \, C\Big(\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds\Big)^3 + \Big(\rho\int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\, ds\Big)^3 + C\\
\leq & \, C\Big(\int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + \rho\widetilde{k}_s^2)\, ds\Big)^3 + C\,,
\end{aligned}$$

for a constant $C$ independent of $r \geq r_i$ and $i \in \mathbb{N}$.
Integrating this differential inequality for the quantity $Q_i(r) = \int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + (r - r_i)\widetilde{k}_s^2)\, ds$ in the interval $[r_i, r_i + 2\delta]$ it is easy to see that if $\delta > 0$ is small enough, we have $Q_i(r) \leq C(\delta, Q_i(r_i)) = C\left(\delta, \int_{\widetilde{\gamma}_{r_i}} \widetilde{k}^2\, ds\right) = C(\delta)$, for every $r \in [r_i, r_i + 2\delta]$, as the curves $\widetilde{\gamma}_{r_i}$ have uniformly bounded curvature in $L^2$. Hence, if $r \in [r_i+\delta, r_i+2\delta]$ we have the estimates

$$\int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + \delta\widetilde{k}_s^2)\, ds \leq \int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + (r - r_i)\widetilde{k}_s^2)\, ds \leq C(\delta)$$

which imply

$$\int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\, ds \leq \frac{C(\delta)}{\delta}\,,$$

for every $r \in [r_i + \delta, r_i + 2\delta]$ and a constant $C(\delta)$ independent of $i \in \mathbb{N}$.
We can now find as before a sequence of values $q_i \in [r_i + \delta, r_i + 2\delta]$ such that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}} \left|\widetilde{k} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 ds = 0$$

and

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}}\, ds = \lim_{i\to\infty} \sigma(t(q_i)) = \Sigma > 1\,.$$

As this new sequence of rescaled curves $\widetilde{\gamma}_{q_i}$ also satisfies the length estimate (3.2.6) and has $\widetilde{k}$ and $\widetilde{k}_s$ uniformly bounded in $L^2$, we can extract a subsequence (not relabelled) that converges in $C^2$ to a limit curve which is again the unit circle.
Then, definitely the curves $\widetilde{\gamma}_{q_i}$ have positive curvature, hence they are convex. This means that the same holds for $\gamma_t$ at some time $t \in [0, T)$, which is Grayson's result.

*Remark* 4.5.8. Pushing this analysis a little forward, one can actually prove along the same lines also the $C^\infty$-convergence of the *full* sequence of the rescaled curves to the unit circle, as proved by Gage and Hamilton in [46, 47, 48].

*Remark* 4.5.9. Actually, the $C^1_{\mathrm{loc}}$-convergence to a line in the case $\Sigma = 1$ allows the possibility of avoiding the application of White's theorem. Indeed, the boundedness of the curvature around every $x_0 \in \mathcal{S}$ the follows also by the interior estimates of Ecker and Huisken.

Another way to conclude without White's theorem is to apply the arguments of this section in getting the blow-up limits around *fixed* points and showing, as under the type 1 hypothesis, that there must be at least one point $x_0 \in \mathcal{S}$ such that a subsequence of the rescalings of the evolving curve around $x_0$ converge to a circle.

We remark that the interesting point of this line in proving Grayson's theorem is the fact that we did not distinguish between *type I* and *type II* singularities. Indeed, the curvature of the rescaled curves could be unbounded, but the control in $L^2_{\mathrm{loc}}$ implies the $C^1_{\mathrm{loc}}$-convergence which is sufficient to obtain the smoothness of the limit curve. In higher dimension the uniform control of the mean curvature in $L^2_{\mathrm{loc}}$ is not strong enough to give the $C^1_{\mathrm{loc}}$-convergence of a subsequence of rescaled hypersurfaces, hence, it is difficult to pursue this "unitary" line of analysis in order to get smooth homothetic blow-up limits also for type II singularities.
It is anyway possible to produce a "homothetic" blow-up limit introducing weak definitions of hypersurfaces (varifolds, currents, see [79]); the difficulty is then to show the regularity and the embeddedness of such a limit.
Some very interesting unpublished results in this direction were obtained by Ilmanen in dimension two [79] (which is, in some sense, the critical case), in particular, assuming the mean convexity of the surfaces, it can be shown that the convergence and the blow-up limits are smooth.

All this discussion underlines the variational nature of the arguments (in particular, the monotonicity formula) in the analysis of type I singularities, against the nonvariational point of view (substantially based on the maximum principle) in dealing with type II ones.

# Chapter 5

# Conclusions and Research Directions

From the material of the previous chapters, the analysis of singularity formation and the classification of their asymptotic shape is almost complete for some classes of hypersurfaces. For others it seems difficult and quite far off. We collect here some known facts and we discuss some research directions and related problems.

## 5.1 Curves in the Plane

### 5.1.1 Embedded Curves

A closed, smooth and embedded curve in the plane evolves, remaining embedded, without developing any singularity till it becomes convex. Then it shrinks smoothly to a point in finite time, becoming asymptotically (exponentially fast) round.

*Gage, Hamilton, Grayson* [46, 47, 48, 51].

### 5.1.2 General Curves

A closed, smooth curve immersed in the plane at a type I singularity has a blow-up limit which is a finite superposition of lines, $\mathbb{S}^1$ and Abresch–Langer curves, possibly with multiplicities larger than one.

In the case that the blow-up limit is compact, the curve shrinks to a point at the singular time.

If the singularity is of type II, the only possible blow-up limit arising from the Hamilton's procedure is the grim reaper (possibly with multiplicity larger than one).

*Abresch & Langer, Angenent, Altschuler, Grayson, Hamilton, Huisken, Stone* [1, 3, 5, 14, 15, 16, 51, 62, 67, 115].

C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progress in Mathematics 290,
DOI 10.1007/978-3-0348-0145-4_5, © Springer Basel AG 2011

## 5.2 Hypersurfaces

### 5.2.1 Entire Graphs

The graph of a locally Lipschitz, entire function $u : \mathbb{R}^n \to \mathbb{R}$ has a smooth, global mean curvature flow, remaining a graph, for every positive time.

*Ecker & Huisken* [37, 38].

### 5.2.2 Convex Hypersurfaces

A compact, smooth and convex initial hypersurface becomes immediately *strictly* convex and shrinks smoothly to a point in finite time, becoming asymptotically (exponentially fast) round.

*Huisken* [65].

### 5.2.3 Embedded Mean Convex Hypersurfaces

A compact, smooth, embedded, $n$-dimensional initial hypersurface with $\mathrm{H} \geq 0$ evolves remaining embedded and becomes immediately *strictly* mean convex.

If we have a type I singularity we can produce a blow-up limit which is one among $\mathbb{S}^m \times \mathbb{R}^{n-m}$ for $m \in \{1, \dots, n\}$.

If the singularity is of type II, the only possible blow-up limits arising by Hamilton's (modified) procedure are the products of an $m$-dimensional, strictly convex, translating mean curvature flow with bounded curvature, with a factor $\mathbb{R}^{n-m}$, for $m \in \{2, \dots, n\}$.

Notice that the grim reaper times $\mathbb{R}^{n-1}$ cannot be among the possible blow-up limits (White [122]).

*Hamilton, Huisken, Sinestrari, Stone, White* [62, 67, 73, 74, 115, 122, 123].

### 5.2.4 Two-Convex Hypersurfaces

The class of two-convex hypersurfaces stays in the middle between the classes of mean convex and convex ones. We recall that $M \subset \mathbb{R}^{n+1}$ is (weakly) two-convex if the sum of the two smallest eigenvalues of the second fundamental form is nonnegative at every point (in particular this implies $\mathrm{H} \geq 0$). In dimension two this class clearly coincides with the class of the mean convex surfaces.

This condition, for a smooth, compact, initial hypersurface, is preserved under the mean curvature flow as we saw in Proposition 2.5.10 and actually after some arbitrarily small positive time, there exists a constant $\alpha > 0$ such that $\lambda_1 + \lambda_2 \geq \alpha \mathrm{H}$ at every point of $M$ and for every positive time (here $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ are the eigenvalues of A).

Since this condition is invariant by rescaling, it must also be satisfied by every blow-up limit and this implies that these latter at a type I singularity can only be the sphere $\mathbb{S}^n$ or the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$.

At a type II singularity only a strictly convex, translating blow-up limit $\widetilde{M}$ is possible, since the product of a factor $\mathbb{R}^{n-m}$ with a strictly convex, $m$-dimensional, translating hypersurface $N$ of $\mathbb{R}^{m+1}$, does not satisfy the condition $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 \geq \alpha\widetilde{\mathrm{H}}$. If $m < n-1$ this is obvious; if $m = n-1$ it is a consequence of the fact that the infimum of the ratio of the minimum eigenvalue of the second fundamental form of $N$ with its mean curvature $\lambda^N_{\min}/\mathrm{H}^N$ must be zero, otherwise by Theorem 4.1.8 the hypersurface $N$ of $\mathbb{R}^n$ is compact, hence cannot be translating. Thus, the blow-up limit $\widetilde{M}$ would have $\widetilde{\lambda}_1 = 0$ (because of the flat factor $\mathbb{R}$) and

$$\liminf_{\widetilde{M}} \widetilde{\lambda}_2/\widetilde{\mathrm{H}} = \liminf_{N} \lambda^N_{\min}/\mathrm{H}^N = 0\,,$$

which is clearly in contradiction with $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 \geq \alpha\widetilde{\mathrm{H}}$.

The interest in the special class of two-convex hypersurfaces is related to the possibility, fully exploited by Huisken and Sinestrari [75], to perform a surgery procedure in order to continue the flow after the singular time, analogous to the one introduced by Hamilton [64] for Ricci flow.

*Huisken & Sinestrari* [75].

### 5.2.5 General Hypersurfaces

About the evolution of a generic compact, smooth, $n$-dimensional, initial hypersurface, we can only say that if it is initially embedded, it stays embedded; if it develops a type I singularity we can produce a (possibly flat) homothetically hypersurface as a blow-up limit and if the singularity is of type II then Hamilton's procedure gives a blow-up limit which is an eternal flow with bounded curvature, such that $|\mathrm{A}|$ achieves its absolute maximum at some point in space and time.

For some special classes of hypersurfaces it is possible to reduce the family of possible blow-up limits at a singularity. For instance, starshaped hypersurfaces [108] or rotationally symmetric ones [4].

*Angenent, Ecker, Huisken, Ilmanen, White* [17, 35, 38, 67, 78, 79, 120, 122, 123].

## 5.3 Mean Curvature Flow with Surgeries

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth, compact, two-convex hypersurface, with $n \geq 3$. In this section we sketchily describe the surgery procedure by Huisken and Sinestrari at a singular time of its mean curvature flow and some of its geometric consequences. We suggest consulting the survey [107] for more details. All the results of this section come from the paper [75].

We remark that the analogous results in the case of surfaces in the Euclidean space ($n = 2$) is an open problem, see anyway Colding and Kleiner [30].

We already said that the evolving hypersurface remains two-convex and embedded if $\varphi_0$ was embedded. Now we discuss a couple of results about the properties of the flow of two-convex hypersurfaces which are essential for the surgery procedure.

**Theorem 5.3.1.** *Let $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$, be a family of compact, two-convex hypersurfaces evolving by mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta$ such that*

$$|\lambda_1| \leq \eta \mathrm{H} \Longrightarrow |\lambda_i - \lambda_j| \leq c(n)\eta\mathrm{H} + C_\eta, \qquad \forall i, j > 1$$

*everywhere on $M$ and for every $t \in [0, T)$, where the constant $c(n) > 0$ depends only on the dimension $n \geq 3$. Here $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of the second fundamental form.*

The above inequality is called a *cylindrical estimate* because it shows that, at a point where H is large and $\lambda_1/\mathrm{H}$ is smaller, the second fundamental form is close to the one of a cylinder, since all the eigenvalues are close to each other with the exception of $\lambda_1$ which is small. Such a property is important because in the surgery procedure one needs to “operate” on regions of the hypersurface which are almost cylinders.

Next, we have the following key inequality for the gradient of the second fundamental form. With respect to the gradient estimates for mean curvature flow already available in the literature, e.g., [28, 38], it should be noticed that this estimate does not depend on the maximum of the curvature in a neighborhood of the point under consideration.

**Theorem 5.3.2.** *Let $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ be a compact, n-dimensional, two-convex hypersurface moving by mean curvature flow, of dimension $n \geq 3$. Then, there exist constants $C = C(n)$ and $D = D(n, \varphi_0)$ such that the flow satisfies the uniform estimate*

$$|\nabla \mathrm{A}|^2 \leq C|\mathrm{A}|^4 + D\,,$$

*for every $t \in [0, T)$.*

Once the estimate for $|\nabla \mathrm{A}|$ is obtained, it is easy to obtain similar estimates for all the higher order derivatives, as well as the time derivatives.

We describe now the construction of mean curvature flow with surgeries, defined in [75], for two-convex hypersurfaces of dimension $n \geq 3$. The aim of performing surgeries is to define a continuation of the flow past the first singular time and all the subsequent ones until the hypersurface vanishes. Another possibility to do this would be to consider weak solutions of the mean curvature flow by the level sets approach, however, weak solutions have generally low regularity past the singular time and it is difficult to analyze the topological changes passing through a singularity. The flow with surgeries is based on a different strategy: if at a singular time $T$ the whole hypersurface vanishes, then we do nothing and consider the flow terminated at time $T$. Otherwise, we consider the flow at some time $t_0$ slightly smaller than $T$ and we cut from the hypersurface $\varphi_{t_0} : M \to \mathbb{R}^{n+1}$ the regions with large curvature, replacing them with less curved ones. Such an operation is called a *surgery*. Possibly, in doing that we could disconnect the hypersurface into several components, in which case we restart the flow independently for each component (or sometimes we simply forget some of them when their topology is trivial) until a new singular time is approached. This procedure is repeated till the vanishing of all these independent components.

The rigorous definition of such a procedure involves a precise knowledge of the geometric properties of the regions that are removed and of the ones that are added as replacements. To this purpose, one introduces the notion of *neck*. The precise definition is given in [75]; roughly speaking, a neck is a portion of a hypersurface which is close, up to a homothety and a rigid motion, to a standard cylinder $[a, b] \times \mathbb{S}^{n-1}$. The surgeries which we consider consist of removing a neck and of replacing it with two regions diffeomorphic to disks which fill smoothly the two holes left at the two ends of the neck. We have then two situations: if the remaining hypersurface is still connected, this topological operation was the inverse of "adding an handle" to the hypersurface (a connected sum with $\mathbb{S}^{n-1} \times \mathbb{S}^1$); if instead the removal of the neck disconnected the hypersurface in two pieces, what we actually performed was the inverse of the connected sum of the two components. If after a finite number of surgeries we know the topology of all the remaining components, keeping track of all the operations we performed, we can reconstruct backward the topology of the initial hypersurface.

This program can be actually carried out and we have the following result.

**Theorem 5.3.3.** *Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a compact, immersed, $n$-dimensional, two-convex, initial hypersurface, with $n \geq 3$. Then there is a mean curvature flow with surgeries such that, after a finite number of surgeries, all the remaining connected components are diffeomorphic either to $\mathbb{S}^n$ or to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

As we said, this theorem implies that the initial manifold can be recovered (up to diffeomorphisms) adding a finite number of handles to the connected sum of a finite family of components all diffeomorphic to $\mathbb{S}^n$ or to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ (all these topological operations are commutative and associative). Recalling that a connected sum with $\mathbb{S}^n$ leaves the topology unchanged, we obtain the following classification of two-convex hypersurfaces.

**Corollary 5.3.4.** *Any smooth, compact, $n$-dimensional, two-convex, immersed hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic either to $\mathbb{S}^n$ or to a finite connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Topological results on $k$-convex hypersurfaces were already known in the literature (see, e.g., [124]). However, these results were based on Morse theory and only ensured homotopic equivalence.

Since the only simply connected hypersurface in the family above is the sphere, another consequence (with some extra arguments) of this surgery procedure based on mean curvature flow is the following Schoenflies type theorem for simply connected, two-convex hypersurfaces.

**Corollary 5.3.5.** *Any smooth, compact, simply connected, $n$-dimensional, two-convex, embedded hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic to $\mathbb{S}^n$ and bounds a region whose closure is diffeomorphic to the $(n+1)$-dimensional, standard closed ball.*

The proof of Theorem 5.3.3 is quite long and technical, we only describe the main points and ideas.

As we said in the previous section, the only possible blow-up limits of the flow of a two-convex hypersurface are the sphere $\mathbb{S}^n$, the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ and the $n$-dimensional, strictly convex, translating hypersurfaces.

When the limit is a sphere, this means that at some time the hypersurface became convex, thus, no surgery is necessary in this case.

If the limit is a cylinder, then we already have the right geometric structure to perform a surgery.

In the case of a translating hypersurface, corresponding to a type II singularity, we have a paraboloid-like hypersurface, but the cylindrical estimates above tell us that a strip of this paraboloid far from the vertex, where the first eigenvalue of the second fundamental form is smaller compared to the others, is actually very close to a strip of a cylinder. Hence, in this case we can choose to perform the surgery not at the point where the curvature takes its maximum, but in a region nearby where the curvature is still quite large. After the surgery, the region containing the vertex of the paraboloid will be thrown away, since it is diffeomorphic to a disc (alternatively, one can "close" such a region in order to have a convex hypersurface that will shrink to a point).

An important point in the surgery procedure is being sure that the curvature actually decreases. To achieve this, one has to choose the necks to be removed in such a way that the "radius" in the central part is much smaller than at the ends. In this way, it is possible to "close" smoothly the holes that are formed in the hypersurface, with two convex *caps* with small curvature.

The above estimates in Theorems 5.3.1 and 5.3.3 (hence the two-convexity, which also restricts the family of the possible blow-up limits) play a fundamental role to prove the existence of necks suitable for the surgery.

To conclude, there are two other essential technical points to be shown in order to make this procedure effective. First, the estimates, with the same constants, must "survive" every surgery. Second, at every step the volume of the hypersurface has to decrease by a fixed positive amount; this clearly implies that the flow necessarily terminates after a finite number of steps, as the volume is decreasing in the time intervals where the flow is smooth.

## 5.4 Some Problems and Research Directions

We mention some problems that are recently receiving attention by the community.

- Noncollapsing results in order to exclude multiplicities in the blow-up limits and to reduce the possible singularity profiles (discussion at the end of Section 4.4).

  See White [122], Ecker [36] and the recent paper by Sheng and Wang [103] where a direct argument by Andrews [10] is also quoted.
- Surgery in dimension two: the extension of the work by Huisken and Sinestrari to the case of mean convex surfaces in $\mathbb{R}^3$ (see Colding and Kleiner [30]) and surgery without the assumption of two-convexity.
- Generic singularities of the flow, that is, showing that the only singularity profiles of the flow for a generic initial hypersurface are spheres or cylinders (a long-standing conjecture of Huisken).

  See Colding and Minicozzi [29].

Finally, we list here some references to research directions related to the mean curvature flow, present in literature. We do not pretend to be exhaustive, we simply want to suggest some starting points for the interested reader.

### 5.4.1 Motion of Noncompact Hypersurfaces

Some very nice results about existence for short time and regularity in the large were obtained by Ecker and Huisken [37, 38], in particular, for graphs of functions and by Chou and Zhu [25] for unbounded curves in the plane.

### 5.4.2 Motion of Hypersurfaces with Boundary

One can consider the mean curvature flow of a hypersurface such that its boundary is fixed or it is forced to have a prescribed angle with another hypersurface, see Huisken [66] or Stahl [112, 113], for instance.

### 5.4.3 Higher Codimension

See the survey of White [121] and, for instance, the global results of Wang [117], the works of Altschuler and Grayson [3, 5] about the evolution of curves in space and the recent paper by Andrews and Baker [11].

### 5.4.4 Evolutions by Different Functions of the Curvature

Instead of taking the mean curvature as the normal speed of the hypersurface, one can consider different functions of the curvature, in particular, any expression in the symmetric functions of the eigenvalues of the second fundamental form, see [8], for instance.

Other possibilities are adding "forcing terms" to a function of the curvature, driving the flow of a hypersurface, or considering evolutions in an ambient space different from the Euclidean space, like a Riemannian manifold.

### 5.4.5 Weak Solutions

The literature on weak formulations of mean curvature flow is quite huge; we simply list some of the main papers that established such weak approaches, see [2, 7, 21, 23, 42, 78].

# Appendix A

# Quasilinear Parabolic Equations on Manifolds

**(In collaboration with Manolo Eminenti and Luca Martinazzi)**

This is the second-order case in the paper [72] by Huisken and Polden (or in the Ph.D. Thesis of Polden [102]).

Let $(M, g)$ be a smooth, compact, $n$-dimensional Riemannian manifold. Let us consider the following PDE problem with a smooth initial datum $u_0 : M \to \mathbb{R}$,

$$\begin{cases} u_t = L(u) & \text{in } M \times [0, T], \\ u(\,\cdot\,, 0) = u_0 & \text{on } M\,, \end{cases} \tag{PDE}$$

where the operator $L$ is a second-order quasilinear differential operator defined in $M \times [0, \mathcal{T})$ (for some $\mathcal{T} > 0$), acting on a function $u : M \times [0, T] \to \mathbb{R}$ with $T < \mathcal{T}$ as follows (in a coordinate chart),

$$L(u) = Q^{ij}(p, t, u, \nabla u)\nabla^2_{ij} u + b(p, t, u, \nabla u)\,,$$

where $Q^{ij}$ and $b$ are smooth functions. Moreover, the operator $L$ is locally elliptic, that is, around every point $p \in M$ there is a coordinate chart such that $Q^{ij} = Q^{ij}(q, s, z, w)$ is a positive definite matrix with lowest eigenvalue uniformly bounded from below away from zero for $(q, s, z, w)$ in some neighborhood of any $(p, t, x, v)$, with $t \in [0, \mathcal{T})$, $x \in \mathbb{R}$ and $v \in T^*_p M$.
It is easy to check that these assumptions on the quasilinear operator are independent of the choice of the coordinate charts.

In order to show the existence of a solution of problem (PDE) in some positive (small) time interval, first we show the existence of a weak solution if the system is linear, then we show its regularity; finally we will deal with the quasilinear case by means of a linearization procedure.


DOI 10.1007/978-3-0348-0145-4, 

## A.1 The Linear Case

Throughout this section we assume that system (PDE) is linear, that is, $L(u) = Q^{ij}\nabla^2_{ij}u + R^k\nabla_k u + Su + b$ and

$$\begin{cases} u_t - Q^{ij}\nabla^2_{ij}u - R^k\nabla_k u - Su = b, \\ u(\,\cdot\,,0) = u_0\,, \end{cases} \tag{A.1.1}$$

where $Q^{ij}$, $R^k$, $S$ and $b$ are smooth functions dependent only on $p \in M$ and on $t \in [0,+\infty)$ (not on the function $u$ or its gradient). Moreover, we suppose that

- the smooth functions $R^k$, $S$, $b$ are bounded in $C^\infty$,
- the functions $Q^{ij}$ are bounded in $C^\infty$ and there exists a uniform ellipticity constant $\lambda > 0$ of $Q^{ij}$.

We set $\widetilde{L}(u) = L(u) - b$. Integrating by parts and using Peter–Paul inequality, we have the following standard Gårding's inequality, for every smooth $u$,

$$-\int_M u\widetilde{L}(u)\,d\mu \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(M)} - C\|u\|^2_{L^2(M)}\,, \tag{A.1.2}$$

for every $t \in [0,+\infty)$, where the constant $C > 0$ depends only on the $C^1$-norms of the functions $Q^{ij}$, $R^k$ and $S$ (see [72] for details).

We define now the function spaces where we will prove the existence of a weak solution of problem (A.1.1).

**Definition A.1.1.** Given $a > 0$, for every pair of functions $f, g \in C^\infty_c(M \times [0,+\infty))$ we set

$$\langle f, g\rangle_{LL_a(M)} = \int_0^{+\infty} e^{-2at}\langle f(\,\cdot\,,t), g(\,\cdot\,,t)\rangle_{L^2(M)}dt\,,$$
$$\langle f, g\rangle_{LW_a(M)} = \int_0^{+\infty} e^{-2at}\langle f(\,\cdot\,,t), g(\,\cdot\,,t)\rangle_{W^{1,2}(M)}dt\,,$$

and let $LL_a(M)$ and $LW_a(M)$ be the Hilbert spaces obtained by completion with respect to the two norms.

Suppose we have a smooth solution $u \in C^\infty(M\times[0,+\infty))$ of problem (A.1.1), then if $\phi \in C^\infty_c(M \times (0,+\infty))$ we have,

$$\begin{aligned} 0 &= \int_0^{+\infty} e^{-2at}\int_M \phi(u_t - L(u))\,d\mu\,dt \\ &= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\ &\quad - \int_0^{+\infty} e^{-2at}\int_M \phi L(u)\,d\mu\,dt \end{aligned}$$

$$\begin{aligned}
&= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\
&\quad - \int_0^{+\infty} e^{-2at}\int_M \phi(Q^{ij}\nabla^2_{ij}u + R^k\nabla_k u + Su + b)\,d\mu\,dt \\
&= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi Su\,d\mu\,dt \\
&\quad - \int_0^{+\infty} e^{-2at}\int_M \phi\, b\,d\mu\,dt\,.
\end{aligned}$$

Notice that, by integration by parts, this equation has a meaning also if $u$ is merely in $LW_a(M)$, so we can use it to define a weak solution of the linear problem.

**Definition A.1.2.** We say that $u \in LW_a(M)$ is a *weak* solution of

$$u_t = Q^{ij}\nabla^2_{ij}u + R^k\nabla_k u + Su + b\,, \tag{A.1.3}$$

if the following equality holds for every $\phi \in C^\infty_c(M \times (0, +\infty))$:

$$\begin{aligned}
&2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\
&+ \int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi Su\,d\mu\,dt \\
&\quad = \int_0^{+\infty} e^{-2at}\int_M \phi\, b\,d\mu\,dt\,.
\end{aligned} \tag{A.1.4}$$

If a weak solution $u$ is smooth it is easy to see that $u$ is a *classical* solution of the parabolic equation in (A.1.1).

We let $\Phi$ be the space of functions $C^\infty_c(M \times (0, +\infty))$ that are clearly zero for small time. If $LW_{a,0}(M)$ is the completion of $\Phi$ with respect to the norm of $LW_a(M)$, asking that a function $u$ belongs to this space is the *weak way* to express the condition $u(\,\cdot\,, 0) = 0$.

By simplicity we define the bilinear form

$$B(u, \phi) = Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi Su\,,$$

and we define $P : LW_{a,0}(M) \times \Phi \to \mathbb{R}$ and $K : \Phi \to \mathbb{R}$ as follows:

$$\begin{aligned}
P(u, \phi) &= 2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(u, \phi)\,d\mu\,dt\,, \\
K(\phi) &= \int_0^{+\infty} e^{-2at}\int_M \phi\, b\,d\mu\,dt\,.
\end{aligned}$$

Then a weak solution $u \in LW_{a,0}(M)$ satisfies $P(u,\phi) = K(\phi)$ for every $\phi \in \Phi$. Notice that $K$ is a continuous functional on $\Phi$ when $b \in LL_a(M)$.

We need now the following variation of Lax–Milgram whose proof can be found in [45, Chapter 10, Theorem 16].

**Lemma A.1.3.** *Let $H$ be a Hilbert space and $\Phi$ a space with a scalar product (not necessarily complete) continuously embedded in $H$. Moreover, let $P : H \times \Phi \to \mathbb{R}$ be a bilinear form such that*

- *$h \mapsto P(h,\phi)$ is continuous for every fixed $\phi \in \Phi$,*
- *$P|_\Phi$ is coercive, that is, there exists a positive constant $C$ such that $P(\phi,\phi) \geq C\|\phi\|^2$, for every $\phi \in \Phi$.*

*Then, for every $K \in \Phi^*$ there exists $h \in H$ such that $K(\phi) = P(h,\phi)$ for every $\phi \in \Phi$.*

**Proposition A.1.4.** *If $b \in LL_a(M)$ and $u_0$ is smooth, the problem (A.1.1) has a weak solution $u \in LW_a(M)$, for $a > 0$ large enough, that is, $u - u_0 \in LW_{a,0}(M)$ and $u$ is a weak solution of equation (A.1.3).*

*Proof.* First we assume that $u_0 = 0$.
For the thesis, it is sufficient to check the hypotheses of Lemma A.1.3. Indeed, it immediately implies the proposition.

Fixing $\phi \in \Phi$, a repeated application of Hölder's inequality shows easily that $L$ and $P$ are continuous with respect to the norm of $LW_a(M)$ (actually, $K$ is already continuous in $LL_a(M)$ like the two first terms of $P(\,\cdot\,,\phi)$).

We show the coerciveness of $P$, keeping in mind that $\phi$ is regular and has compact support in $M \times (0,+\infty)$,

$$\begin{aligned}
P(\phi,\phi) = & \, 2a\int_0^{+\infty} e^{-2at}\int_M \phi^2\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t\phi\,d\mu\,dt \\
& + \int_0^{+\infty} e^{-2at}\int_M B(\phi,\phi)\,d\mu\,dt \\
= & \, 2a\int_0^{+\infty} e^{-2at}\int_M \phi^2\,d\mu\,dt - a\int_0^{+\infty} e^{-2at}\int_M \phi^2\,d\mu\,dt \\
& - \int_0^{+\infty} e^{-2at}\int_M \phi\widetilde{L}(\phi)\,d\mu\,dt \\
\geq & \, a\int_0^{+\infty} e^{-2at}\int_M \phi^2\,d\mu\,dt + \frac{\lambda}{2}\int_0^{+\infty} e^{-2at}\int_M |\nabla\phi|^2\,d\mu\,dt \\
& - C\int_0^{+\infty} e^{-2at}\int_M \phi^2\,d\mu\,dt\,,
\end{aligned}$$

where we used Gårding's inequality (A.1.2), hence

$$P(\phi,\phi) \geq (a-C)\|\phi\|^2_{LL_a(M)} + \frac{\lambda}{2}\|\phi\|^2_{LW_a(M)}\,.$$

Choosing $a > C$, the coerciveness of $P$ follows.

Suppose now that the smooth initial datum $u_0$ is not identically zero. We consider the equation satisfied by the function $v = u - u_0$, with a null initial datum and we solve it with the previous method.
The last term of this new problem is $L(u_0) = \widetilde{L}(u_0) + b$ (the other parts of the operator are the same, by linearity), then the regularity of $u_0$ implies that it satisfies the hypotheses for the existence of a weak solution $v \in LW_{a,0}(M)$, hence of $u \in LW_a(M)$ as in the statement of the proposition. $\square$

**Lemma A.1.5.** *If $u \in LW_a(M)$ is a weak solution of problem* (A.1.1)*, then for every $\phi \in C_c^\infty(M \times [0, +\infty))$, the following equation holds:*

$$\begin{aligned}
&2a \int_0^{+\infty} e^{-2at} \int_M \phi u \, d\mu \, dt - \int_0^{+\infty} e^{-2at} \int_M \phi_t u \, d\mu \, dt - \int_M \phi u_0 \, d\mu \,, \qquad \text{(A.1.5)} \\
&+ \int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i \phi \nabla_j u + \phi \nabla_j u \nabla_i Q^{ij} - \phi R^k \nabla_k u - \phi S u \, d\mu \, dt \\
&= \int_0^{+\infty} e^{-2at} \int_M \phi \, b \, d\mu \, dt \,.
\end{aligned}$$

*Proof.* Let $v = u - u_0 \in LW_{a,0}(M)$ satisfying the relative equation (A.1.4) in the modified system, that is,

$$\begin{aligned}
&2a \int_0^{+\infty} e^{-2at} \int_M \phi v \, d\mu \, dt - \int_0^{+\infty} e^{-2at} \int_M \phi_t v \, d\mu \, dt \qquad \text{(A.1.6)} \\
&+ \int_0^{+\infty} e^{-2at} \int_M Q^{ij} \nabla_i \phi \nabla_j v + \phi \nabla_j v \nabla_i Q^{ij} - \phi R^k \nabla_k v - \phi S v \, d\mu \, dt \\
&= \int_0^{+\infty} e^{-2at} \int_M \phi \, L(u_0) + b \, d\mu \, dt \,,
\end{aligned}$$

for every function $\phi \in C_c^\infty(M \times (0, +\infty))$.
Let now $\phi = \sigma\varphi$ where $\varphi \in C_c^\infty(M \times [0, +\infty))$ and $\sigma : [0 + \infty) \to \mathbb{R}$ is a function which is zero on $[0, \varepsilon]$, one on $[2\varepsilon, +\infty)$ and linear in the middle (we can "put" such a function $\sigma$ in the formula above by approximation).
We compute,

$$\begin{aligned}
&2a \int_0^{+\infty} e^{-2at} \sigma \int_M \varphi v \, d\mu \, dt - \int_0^{+\infty} e^{-2at} \sigma \int_M \varphi_t v \, d\mu \, dt \\
&- \frac{1}{\varepsilon} \int_\varepsilon^{2\varepsilon} e^{-2at} \int_M \varphi v \, d\mu \, dt \\
&+ \int_0^{+\infty} e^{-2at} \sigma \int_M Q^{ij} \nabla_i \varphi \nabla_j v + \phi \nabla_j v \nabla_i Q^{ij} - \phi R^k \nabla_k v - \phi S v \, d\mu \, dt \\
&= \int_0^{+\infty} e^{-2at} \sigma \int_M \varphi \, L(u_0) + b \, d\mu \, dt \,.
\end{aligned}$$

As $\varepsilon \to 0$, the term $\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon} e^{-2at}\int_M \varphi v\,d\mu\,dt$ converges to zero as $v \in LW_{a,0}(M)$ and the other terms converge to the corresponding ones without $\sigma$ inside, as $\sigma \to \chi_{[0,+\infty)}$ when $\varepsilon \to 0$. Hence, $v$ satisfies relation (A.1.6) for every $\phi \in C^\infty_c(M \times [0,+\infty))$.
Substituting $v = u - u_0$ we get

$$\begin{aligned}
&2a\int_0^{+\infty} e^{-2at}\int_M \phi u\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M \phi_t u\,d\mu\,dt\\
&-2a\int_0^{+\infty} e^{-2at}\int_M \phi u_0\,d\mu\,dt + \int_0^{+\infty} e^{-2at}\int_M \phi_t u_0\,d\mu\,dt\\
&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi S u\,d\mu\,dt\\
&\quad=\int_0^{+\infty} e^{-2at}\int_M \phi\, b\,d\mu\,dt\,,
\end{aligned}$$

and the second line is equal to

$$\begin{aligned}
-2a\int_0^{+\infty} e^{-2at}\int_M \phi u_0\,d\mu\,dt + \int_0^{+\infty} e^{-2at}\int_M \phi_t u_0\,d\mu\,dt
&= -2a\int_0^{+\infty} e^{-2at}\int_M \phi u_0\,d\mu\,dt\\
&\quad+\int_0^{+\infty} e^{-2at}\frac{d}{dt}\int_M \phi u_0\,d\mu\,dt\\
&=\int_0^{+\infty}\frac{d}{dt}\Big[e^{-2at}\int_M \phi u_0\,d\mu\Big]\,dt\\
&=-\int_M \phi u_0\,d\mu\,,
\end{aligned}$$

which gives the thesis of the lemma. □

## A.2 Regularity in the Linear Case

**Definition A.2.1.** Let

$$LW^s_a(M) = \left\{ f : M \times [0,+\infty) \to \mathbb{R} \;\middle|\; \int_0^{+\infty} e^{-2at}\|f(\,\cdot\,,t)\|^2_{W^{s,2}(M)}\,dt < +\infty \right\}$$

with the scalar product

$$\langle f, g\rangle_{LW^s_a(M)} = \int_0^{+\infty} e^{-2at}\langle f(\,\cdot\,,t), g(\,\cdot\,,t)\rangle_{W^{s,2}(M)}\,dt\,.$$

Moreover,

$$P^l_a(M) = \left\{ f : M \times [0, +\infty) \to \mathbb{R} \,\middle|\, \frac{\partial^j f}{\partial t^j} \in LW^{2(l-j)}_a(M) \ \forall j \leq l \right\},$$

where $\frac{\partial^j f}{\partial t^j}$ is in distributional sense. Clearly $P^l_a(M) \subset LW^{2l}_a(M)$.
On this space we have the scalar product

$$\langle f, g\rangle_{P^l_a(M)} = \sum_{j\leq l} \left\langle \frac{\partial^j f}{\partial t^j}, \frac{\partial^j g}{\partial t^j} \right\rangle_{LW^{2(l-j)}_a(M)} .$$

**Lemma A.2.2.** *The trace $f_0$ of a function $f \in P^l_a(M)$ on the parabolic boundary $M \times \{0\}$ belongs to the space $W^{2l-1,2}(M)$.*

*Proof.* It is easy to see that $C^\infty_c(M \times [0, +\infty))$ is dense in $P^l_a(M)$. If $f$ is smooth with compact support, we have

$$\begin{aligned}
\int_0^{+\infty} e^{-2at} \int_M g\Big(\nabla^{2l-1} f, \nabla^{2l-1} \frac{\partial f}{\partial t}\Big)\, d\mu\, dt &= \frac{1}{2} \int_0^{+\infty} e^{-2at} \int_M \frac{\partial}{\partial t} |\nabla^{2l-1} f|^2\, d\mu\, dt \\
&= a \int_0^{+\infty} e^{-2at} \int_M |\nabla^{2l-1} f|^2\, d\mu\, dt \\
&\quad - \frac{1}{2} \int_M |\nabla^{2l-1} f_0|^2\, d\mu\,.
\end{aligned}$$

Hence, keeping in mind the definition of the space $P^l_a(M)$,

$$\begin{aligned}
\int_M |\nabla^{2l-1} f_0|^2\, d\mu &= 2a \int_0^{+\infty} e^{-2at} \int_M |\nabla^{2l-1} f|^2\, d\mu\, dt \\
&\quad - 2 \int_0^{+\infty} e^{-2at} \int_M g\Big(\nabla^{2l-1} f, \nabla^{2l-1} \frac{\partial f}{\partial t}\Big)\, d\mu\, dt \\
&= 2a \int_0^{+\infty} e^{-2at} \int_M |\nabla^{2l-1} f|^2\, d\mu\, dt \\
&\quad + 2 \int_0^{+\infty} e^{-2at} \int_M g\Big(\Delta\nabla^{2l-2} f, \nabla^{2l-2} \frac{\partial f}{\partial t}\Big)\, d\mu\, dt \\
&\leq 2a\|f\|^2_{LW^{2l-1}_a(M)} + 2\|f\|_{LW^{2l}_a(M)}\|f\|_{P^l_a(M)} \\
&\leq 3a\|f\|^2_{P^l_a(M)} .
\end{aligned}$$

Then, the conclusion follows by approximation. □

We are now ready to state the main result of this section.

**Proposition A.2.3.** *For every $l \in \mathbb{N}$ the linear map*

$$F(u) = \Big(u_0, u_t - \widetilde{L}(u)\Big) \tag{A.2.1}$$

*is an isomorphism of $P^l_a(M)$ onto $W^{2l-1,2}(M) \times P^{l-1}_a(M)$, for $a > 0$ large enough.*

By the previous lemma the map $F$ is well defined and bounded, we only need to show that it is a bijection, by the open mapping theorem.

**Lemma A.2.4.** *If $u \in LW_a(M)$ satisfies equation* (A.1.5) *for every $\phi \in C_c^\infty(M \times [0,+\infty))$ then, the following estimate holds:*

$$\|u\|^2_{LW_a(M)} \le C\left(\|u_0\|^2_{L^2(M)} + \|b\|^2_{LL_a(M)}\right).$$

*Proof.* Let $\phi \in C_c^\infty(M\times[0,+\infty))$, by looking at equation (A.1.5) we can see easily that $u_t \in LL_a(M)$. Then, by the density of $C_c^\infty(M \times [0,+\infty))$ in $LW_a(M)$), we can substitute $\phi$ with $u$, obtaining

$$\begin{aligned}
&2a\int_0^{+\infty} e^{-2at}\int_M u^2\,d\mu\,dt - \int_0^{+\infty} e^{-2at}\int_M u_t u\,d\mu\,dt - \int_M u_0^2\,d\mu\\
&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i u\nabla_j u + u\nabla_j u\nabla_i Q^{ij} - uR^k\nabla_k u - Su^2\,d\mu\,dt\\
&\quad=\int_0^{+\infty} e^{-2at}\int_M u\,b\,d\mu\,dt\,.
\end{aligned}$$

Taking the time derivative outside the inner integral of the second term and integrating by parts, we get

$$\begin{aligned}
&a\int_0^{+\infty} e^{-2at}\varphi\int_M u^2\,d\mu\,dt - \frac{1}{2}\int_M u_0^2\,d\mu\\
&+\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i u\nabla_j u + u\nabla_j u\nabla_i Q^{ij} - uR^k\nabla_k u - Su^2\,d\mu\,dt\\
&\quad=\int_0^{+\infty} e^{-2at}\int_M u\,b\,d\mu\,dt\,.
\end{aligned}$$

By Gårding's inequality (A.1.2) this formula implies

$$\frac{\lambda}{2}\|u\|^2_{LW_a(M)} \le (C-a)\|u\|^2_{LL_a(M)} + \frac{1}{2}\|u_0\|^2_{L^2(M)} + \|u\|_{LL_a(M)}\|b\|_{LL_a(M)}$$

and using Peter–Paul inequality on the last term,

$$\|b\|_{LL_a(M)}\|u\|_{LL_a(M)} \le \varepsilon\|u\|^2_{LL_a(M)} + C\|b\|^2_{LL_a(M)} \le \varepsilon\|u\|^2_{LW_a(M)} + C\|b\|^2_{LL_a(M)},$$

the lemma follows by choosing $a > C$ and $\varepsilon < \lambda/4$. □

In order to get estimates on the higher derivatives of $u$ we work with the incremental ratios, being the ambient space $M$ a manifold we need to use local charts.
Given $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h \neq 0$, fixing $v \in R^n$, let

$$(\Theta_h f)(x) = h^{-1}(f(x+hv) - f(x))\,.$$

The following properties of the operators $\Theta_h$ are easily checked.

- If $f, g : \mathbb{R}^n \to \mathbb{R}^n$ then

$$(\Theta_h(fg))(x) = (\Theta_h f)(x)g(x+hv) + (\Theta_h g)(x)f(x)\,.$$

- If $f, g \in L^1(\mathbb{R}^n)$ with compact support contained in an open set $\Omega \subset \mathbb{R}^n$, then for $h$ small enough we have

$$\int_\Omega \Theta_h f\,dx = 0$$

and

$$\int_\Omega f\,\Theta_h g\,dx = -\int_\Omega g\,\Theta_{-h} f\,dx\,.$$

Let $\psi_l : \mathbb{R}^n \supset B^n_1 \to M$, for $l = 1, \dots, m$, be a family of local charts such that the union of $\psi_l(B^n_{1/2})$ covers $M$ ($B^n_r$ is the $n$-dimensional ball of radius $r$). Moreover, let $\rho : \mathbb{R}^n \to [0,1]$ be a smooth function which is 0 outside $B^n_{3/4}$ and 1 in $B^n_{1/2}$. We define $U_l = \psi_l(B^n_1)$ and $V_l = \psi_l(B^n_{1/2})$.

We "lift" now $\Theta$ and $\rho$ on $M$ via the coordinate charts $\psi_l$, still using the same notation. Notice that the relations above still hold for functions on $M$ whose support is contained in a single chart.

**Lemma A.2.5.** *If $u \in LW_a(M)$ satisfies equation* (A.1.5) *with a smooth $u_0$, then $u \in LW^2_a(M)$ and the following estimate holds:*

$$\|u\|^2_{LW^2_a(M)} \le C\left(\|u_0\|^2_{W^{1,2}(M)} + \|b\|^2_{LL_a(M)}\right). \tag{A.2.2}$$

*Proof.* We prove the estimate

$$\|\nabla^2 u\|^2_{LL_a(M)} \le C\left(\|u_0\|^2_{W^{1,2}(M)} + \|u\|^2_{LW_a(M)} + \|b\|^2_{LL_a(M)}\right), \tag{A.2.3}$$

then the conclusion follows by means of Lemma A.2.4.
We fix a chart $\psi_l$ and we consider the test function $\phi = \Theta_{-h}(\rho^2\Theta_h u)$ extended to 0 outside $U_l$ in equation (A.1.5); integrating by parts, as $u_t \in LL_a(M)$, we get

$$\begin{aligned}\left\langle u_t, \Theta_{-h}(\rho^2\Theta_h u)\right\rangle_{LL_a(M)} + \int_0^{+\infty} e^{-2at}\int_M B(u, \Theta_{-h}(\rho^2\Theta_h u))\,d\mu\,dt \\ = \langle b, \Theta_{-h}(\rho^2\Theta_h u)\rangle_{LL_a(M)}\,,\end{aligned}$$

recalling that we defined the form $B$ as

$$B(u,\phi) = Q^{ij}\nabla_i\phi\nabla_j u + \phi\nabla_j u\nabla_i Q^{ij} - \phi R^k\nabla_k u - \phi S u\,.$$

Now "moving" the incremental ratios and integrating by parts, we obtain

$$
\begin{aligned}
-\langle b, \Theta_{-h}(\rho^2\Theta_h u)\rangle_{LL_a(M)} &= \left\langle \frac{\partial(\Theta_h u)}{\partial t}, \rho^2\Theta_h u\right\rangle_{LL_a(M)} \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt \\
&= \left\langle \frac{\partial(\rho\Theta_h u)}{\partial t}, \rho\Theta_h u\right\rangle_{LL_a(M)} \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt \\
&= a\int_0^{+\infty} e^{-2at}\int_M \rho^2|\Theta_h u|^2\,d\mu\,dt - \frac{1}{2}\int_M \rho^2|\Theta_h u_0|^2\,d\mu \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\quad + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt
\end{aligned}
$$

where the term $(\Theta_h B)(u, \rho^2\Theta_h u)$ is given by the application of the Leibniz rule for the incremental ratios. Anyway, this term is not a problem as all coefficients of the form $B$ are bounded in $C^\infty$.

Hence, we have

$$
\begin{aligned}
&\int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt + \int_0^{+\infty} e^{-2at}\int_M (\Theta_h B)(u, \rho^2\Theta_h u)\,d\mu\,dt \\
&\qquad \leq C\|b\|_{LL_a(M)}\|\Theta_{-h}(\rho^2\Theta_h u)\|_{LL_a(M)} + C\|\Theta_h u\|^2_{LL_a(U_l)} + \frac{1}{2}\int_M \rho^2|\Theta_h u_0|^2\,d\mu \\
&\qquad \leq C\|b\|_{LL_a(M)}\|\rho^2\Theta_h u\|_{LW_a(M)} + C\|u\|^2_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)}\,,
\end{aligned}
$$

by the standard integral estimates on the incremental ratios.

We deal now with the two integrals in this formula. For the first one, which we call $\mathcal{A}$, after some manipulations with the Leibniz rule, we have

$$
\mathcal{A} = \int_0^{+\infty} e^{-2at}\int_M B(\Theta_h u, \rho^2\Theta_h u)\,d\mu\,dt
$$

$$
\begin{aligned}
&\geq \int_0^{+\infty} e^{-2at}\int_M \rho Q^{ij}\nabla_i\Theta_h u\nabla_j(\rho\Theta_h u)\,d\mu\,dt\\
&\quad - C\int_0^{+\infty} e^{-2at}\int_M \rho|\nabla\Theta_h u|\,|\Theta_h u|\,|\nabla\rho|\,d\mu\,dt\\
&\quad - C\int_0^{+\infty} e^{-2at}\int_M \rho^2|\Theta_h u|(|\nabla\Theta_h u|+|\Theta_h u|)\,d\mu\,dt\\
&\geq \int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i(\rho\Theta_h u)\nabla_j(\rho\Theta_h u)\,d\mu\,dt\\
&\quad - C\int_0^{+\infty} e^{-2at}\int_M |\nabla(\rho\Theta_h u)|\,|\Theta_h u|\,(\rho+|\nabla\rho|)\,d\mu\,dt\\
&\quad - C\int_0^{+\infty} e^{-2at}\int_M |\Theta_h u|^2(\rho^2+\rho|\nabla\rho|+|\nabla\rho|^2)\,d\mu\,dt\\
&\geq \int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i(\rho\Theta_h u)\nabla_j(\rho\Theta_h u)\,d\mu\,dt\\
&\quad - \varepsilon_l\int_0^{+\infty} e^{-2at}\int_M |\nabla(\rho\Theta_h u)|^2\,d\mu\,dt\\
&\quad - C_l\int_0^{+\infty} e^{-2at}\int_{U_l} |\Theta_h u|^2\,d\mu\,dt\\
&\geq \int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i(\rho\Theta_h u)\nabla_j(\rho\Theta_h u)\,d\mu\,dt\\
&\quad - \varepsilon_l\|\nabla(\rho\Theta_h u)\|^2_{LL_a(M)} - C_l\|\Theta_h u\|^2_{LL_a(U_l)}\,,
\end{aligned}
$$

for some small $\varepsilon_l > 0$ and constants $C_l$ that we obtained by means of the use of Peter–Paul inequality. Again, by standard estimates, we conclude

$$
\begin{aligned}
\mathcal{A} \geq &\int_0^{+\infty} e^{-2at}\int_M Q^{ij}\nabla_i(\rho\Theta_h u)\nabla_j(\rho\Theta_h u)\,d\mu\,dt\\
&- \varepsilon_l\|\rho\Theta_h u\|^2_{LW_a(M)} - C_l\|u\|^2_{LW_a(M)}\,,
\end{aligned}
$$

where the constants $\varepsilon_l$ and $C_l$ are independent of $u$ and $h$ (small enough). By Gårding's inequality (A.1.2) it follows easily that

$$
\begin{aligned}
\int_0^{+\infty} & e^{-2at}\int_M Q^{ij}\nabla_i(\rho\Theta_h u)\nabla_j(\rho\Theta_h u)\,d\mu\,dt\\
&\geq \frac{\lambda}{2}\|\rho\Theta_h u\|^2_{LW_a(M)} - C\|\rho\Theta_h u\|^2_{LL_a(M)}\\
&\geq \frac{\lambda}{2}\|\rho\Theta_h u\|^2_{LW_a(M)} - C\|u\|^2_{LW_a(M)}\,,
\end{aligned}
$$

hence,

$$
\mathcal{A} \geq \Big(\frac{\lambda}{2} - \varepsilon_l\Big)\|\rho\Theta_h u\|^2_{LW_a(M)} - C_l\|u\|^2_{LW_a(M)}\,.
$$

For the second integral, which we call $\mathcal{B}$, we have

$$
\begin{aligned}
\mathcal{B} = &\int_0^{+\infty} e^{-2at} \int_M (\Theta_h B)(u, \rho^2 \Theta_h u)\, d\mu\, dt \\
\geq &\int_0^{+\infty} e^{-2at} \int_M (\Theta_h Q^{ij}) \nabla_i u \nabla_j (\rho^2 \Theta_h u)\, d\mu\, dt \\
&- C\int_0^{+\infty} e^{-2at} \int_M \rho^2 |\Theta_h u|\, (|\nabla u| + |u|)\, d\mu\, dt \\
\geq &- C\int_0^{+\infty} e^{-2at} \int_M \rho |\nabla u|\, |\nabla(\rho \Theta_h u)|\, d\mu\, dt \\
&- C\int_0^{+\infty} e^{-2at} \int_M \rho |\Theta_h u|\, |\nabla u|\, |\nabla \rho|\, d\mu\, dt \\
&- C\int_0^{+\infty} e^{-2at} \int_M \rho^2 |\Theta_h u|\, (|\nabla u| + |u|)\, d\mu\, dt \\
\geq &- \varepsilon_l \|\rho \Theta_h u\|^2_{LW_a(M)} - C_l \|u\|^2_{LW_a(M)}\,,
\end{aligned}
$$

by repeated use of Peter–Paul inequality.
Finally we obtain

$$
\begin{aligned}
\Big(\frac{\lambda}{2} - \varepsilon_l\Big) \|\rho \Theta_h u\|^2_{LW_a(M)} \leq\, & \mathcal{A} + C_l \|u\|^2_{LW_a(M)} \\
\leq\, & C\|b\|_{LL_a(M)} \|\rho^2 \Theta_h u\|_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} \\
& - \mathcal{B} + C_l \|u\|^2_{LW_a(M)} \\
\leq\, & C\|b\|_{LL_a(M)} \|\rho^2 \Theta_h u\|_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} \\
& + \varepsilon_l \|\rho \Theta_h u\|^2_{LW_a(M)} + C_l \|u\|^2_{LW_a(M)} \\
\leq\, & C_l \|b\|^2_{LL_a(M)} + \varepsilon_l \|\rho^2 \Theta_h u\|^2_{LW_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} \\
& + \varepsilon_l \|\rho \Theta_h u\|^2_{LW_a(M)} + C_l \|u\|^2_{LW_a(M)}\,.
\end{aligned}
$$

It can be seen easily that

$$
\begin{aligned}
\|\rho^2 \Theta_h u\|^2_{LW_a(M)} &\leq \|\rho \Theta_h u\|^2_{LW_a(M)} + C\|\Theta_h u\|^2_{LL_a(U_l)} \\
&\leq \|\rho \Theta_h u\|^2_{LW_a(M)} + C\|u\|^2_{LW_a(M)}\,,
\end{aligned}
$$

as $\rho \leq 1$, hence,

$$
\Big(\frac{\lambda}{2} - 3\varepsilon_l\Big) \|\rho \Theta_h u\|^2_{LW_a(M)} \leq C_l \|b\|^2_{LL_a(M)} + C\|u_0\|^2_{W^{1,2}(M)} + C_l \|u\|^2_{LW_a(M)}\,.
$$

By the arbitrariness of $h$, after choosing $\varepsilon_l$ small enough, this estimate implies that

$$
\|\nabla^2 u\|^2_{LL_a(V_l)} \leq C_l \Big(\|b\|^2_{LL_a(M)} + \|u_0\|^2_{W^{1,2}(M)} + \|u\|^2_{LW_a(M)}\Big)\,.
$$

As the coordinate charts are finite ($M$ is compact) we get inequality (A.2.3), concluding the proof. □

An immediate consequence of the fact that $u \in LW^2_a(M)$ is the estimate

$$\begin{aligned}\|u_t\|^2_{LL_a(M)} &\leq C\Big(\|u\|^2_{LW^2_a(M)} + \|u_0\|^2_{LL_a(M)} + \|b\|^2_{LL_a(M)}\Big)\\ &\leq C\Big(\|u_0\|^2_{W^{1,2}(M)} + \|b\|^2_{LL_a(M)}\Big),\end{aligned}$$

that follows from equation (A.1.5).
Combining it with estimate (A.2.2) we have the following proposition.

**Proposition A.2.6.** *If $u_0 \in W^{1,2}(M)$ and $b \in LL_a(M)$ then $u \in P^1_a(M)$, with the estimate*

$$\|u\|^2_{P^1_a} \leq C\left(\|u_0\|^2_{W^{1,2}(M)} + \|b\|^2_{LL_a(M)}\right).$$

*Remark* A.2.7. By means of approximation with smooth functions, this proposition implies the existence and uniqueness of a weak solution of problem (A.1.1) with an initial datum $u_0 \in W^{1,2}(M)$.

Suppose now that $b \in P^{l-1}_a(M)$ (hence, $b \in LW^{2l-2}_a(M)$) and $u_0 \in W^{2l-1,2}(M)$. We consider the test function $\Theta^{2l-1}_{-h}\rho^2\Theta^{2l-1}_h u$ and we work as in the proof of Lemma A.2.5.

$$\begin{aligned}&-\langle b, \Theta^{2l-1}_{-h}(\rho^2\Theta^{2l-1}_h u)\rangle_{LL_a(M)}\\ &\quad= \left\langle \frac{\partial(\Theta^{2l-1}_h u)}{\partial_t}, \rho^2\Theta^{2l-1}_h u\right\rangle_{LL_a(M)}\\ &\qquad+ \int_0^{+\infty} e^{-2at}\int_M B(\Theta^{2l-1}_h u, \rho^2\Theta^{2l-1}_h u)\,dt\\ &\qquad+ \sum_{j=1}^{2l-1}\binom{2l-1}{j}(-1)^j\int_0^{+\infty} e^{-2at}\int_M (\Theta^j_h B)(\Theta^{2l-1-j}_h u, \rho^2\Theta^{2l-1}_h u)\,dt\,.\end{aligned}$$

Proceeding analogously, with the only difference that we deal with the term containing $b$ as follows,

$$\begin{aligned}-\langle b, \Theta^{2l-1}_{-h}(\rho^2\Theta^{2l-1}_h u)\rangle_{LL_a(M)} &\leq |\langle \Theta^{2l-2}_h b, \Theta_{-h}(\rho^2\Theta^{2l-1}_h u)\rangle_{LL_a(M)}|\\ &\leq \|b\|_{LW^{2l-2}_a(M)}\|\rho^2\Theta^{2l-1}_h u\|_{LW_a(M)}\,,\end{aligned}$$

we obtain

$$\|\nabla(\rho\Theta^{2l-1}_h u)\|^2_{LL_a(M)} \leq C\Big(\|u_0\|^2_{W^{2l-1,2}(M)} + \|u\|^2_{LW^{2l-1}_a(M)} + \|b\|^2_{LW^{2l-2}_a(M)}\Big).$$

By means of Proposition A.2.6 and iteration, we conclude

$$\|u\|^2_{LW^{2(l-1)}_a(M)} \leq C\Big(\|u_0\|^2_{W^{2l-1,2}(M)} + \|b\|^2_{LW^{2l-2}_a(M)}\Big).$$

Suppose now, by induction, that for every $j < m \leq l$ we have $\frac{\partial^j u}{\partial t^j} \in LW^{2(l-1-j)}_a(M)$, the case $j = 0$ being the previous estimate. Putting in equation (A.1.5) a smooth test function $\phi = \frac{\partial^{m-1}}{\partial t^{m-1}}\varphi$ in $C^\infty_c(M\times(0,+\infty))$, integrating by parts and estimating, we see that $u$ satisfies the estimates

$$\begin{aligned}\left\|\frac{\partial^m u}{\partial t^m}\right\|^2_{LW_a^{2(l-1-m)}(M)} &\leq C\Big(\left\|\frac{\partial^{m-1} b}{\partial t^{m-1}}\right\|^2_{LW_a^{2(l-1-m)}(M)} + \sum_{j<m}\left\|\frac{\partial^j u}{\partial t^j}\right\|^2_{LW_a^{2(l-1-j)}(M)}\Big)\\ &\leq C\Big(\|b\|^2_{P_a^{l-1}(M)} + \sum_{j<m}\left\|\frac{\partial^j u}{\partial t^j}\right\|^2_{LW_a^{2(l-1-j)}(M)}\Big).\end{aligned}$$

Hence, we conclude that $\frac{\partial^j u}{\partial t^j} \in LW_a^{2(l-1-j)}(M)$, for every $j\in\{0,\dots,l\}$.
We summarize all this argument in the following proposition.

**Proposition A.2.8.** *For every $l\in\mathbb{N}$, if $u_0 \in W^{2l-1,2}(M)$ and $b\in P_a^{l-1}(M)$ then $u\in P_a^l(M)$ with the estimate*

$$\|u\|^2_{P_a^l(M)} \leq C\left(\|u_0\|^2_{W^{2l-1,2}(M)} + \|b\|^2_{P_a^{l-1}(M)}\right).$$

We can now show Proposition A.2.3.

*Proof of Proposition A.2.3.* As we already said, the map $F$ is well defined and continuous and in order to conclude that $F$ is an isomorphism it is sufficient to show that there exists a unique weak solution in $u\in P_a^l(M)$ of problem (A.1.1) with any initial datum $u_0\in W^{2l-1,2}(M)$ and $b\in P_a^{l-1}(M)$.
As $W^{2l-1,2}(M)$ is a subspace of $W^{1,2}(M)$, by Remark A.2.7 we have a unique weak solution $u\in LW_a(M)$, then Proposition A.2.8 implies that $u\in P_a^l(M)$, as we wanted. □

## A.3 The General Case

**Theorem A.3.1.** *Problem* (PDE) *has a unique smooth solution defined in a time interval $[0,T)$ and such solution depends continuously in $C^\infty$ on the smooth initial datum $u_0$.*

As $M$ is compact there exists a constant $C>0$ such that the initial datum satisfies $|u_0|+|\nabla u_0|_g \leq C$. Since we are looking for a short time solution, possibly modifying the functions $Q^{ij}$ and $b$ with some “cut-off” functions, we can assume that if $|u|+|\nabla u|_g + t \geq 2C$, the matrix $Q^{ij}(p,t,u,\nabla u)$ coincides with $g^{ij}(p)$ and $b(p,t,u,\nabla u)$ is zero. It follows that the operator $Q^{ij}(\,\cdot\,)\nabla^2_{ij}$ has an ellipticity constant $\lambda>0$ uniformly bounded from below away from zero and that for large time there holds $L(u)=\Delta u$.

For any $l\in\mathbb{N}$ we define $P^l(M,T)$ as the completion of $C^\infty(M\times[0,T])$ under the norm

$$\|f\|^2_{P^l(M,T)} = \sum_{j,\,k\,\in\,\mathbb{N}\text{ and }2j+k\,\leq\,2l}\int_{M\times[0,T]} |\partial_t^j\nabla^k f|^2\,d\mu\,dt\,,$$

for every $T\in\mathbb{R}^+$.

Clearly, there is a natural continuous embedding $P^l_a(M) \hookrightarrow P^l(M,T)$. In the following it will be easier (though conceptually equivalent) to use the spaces $P^l(M,T)$ instead of Polden's weighted spaces $P^l_a(M)$. For this reason we translate Proposition A.2.3 into the setting of $P^l(M,T)$ spaces.

**Proposition A.3.2.** *For every $T>0$ and $l\in\mathbb{N}$ the map $F$ defined by formula* (A.2.1) *is an isomorphism of $P^l(M,T)$ onto $W^{2l-1,2}(M)\times P^{l-1}(M,T)$.*

*Proof.* The continuity of the second component of $F$ is obvious while the continuity of the first component follows by arguing as in the trace Lemma A.2.2. Hence, the map $F$ is continuous, now we show that it is an isomorphism.

Given any $b\in P^{l-1}(M,T)$ we consider an extension $\widetilde{b}\in P^{l-1}_a(M)$ of the function $b$ and we let $\widetilde{u}\in P^l_a(M)$ be the solution of problem (A.1.1) for $\widetilde{b}$. Clearly, $u=\widetilde{u}|_{M\times[0,T]}$ belongs to $P^l(M,T)$ and satisfies $F(u)=(u_0,b)$ in $M\times[0,T]$. Suppose that $v\in P^m(M,T)$ is another function such that $F(v)=(u_0,b)$ in $M\times[0,T]$, then by the maximum principle the functions $u$ and $v$ must coincide in all $M\times[0,T]$.

Since the map $F:P^l(M,T)\to W^{2l-1,2}(M)\times P^{l-1}(M,T)$ is continuous, one-to-one and onto, it is an isomorphism by the open mapping theorem. □

*Remark* A.3.3. When $u_0$ and $b$ are smooth the unique solution $u$ of problem (A.1.1) belongs to all the spaces $P^l(M,T)$ for every $l\in\mathbb{N}$. As by Sobolev embeddings for any $k\in\mathbb{N}$ we can find a large $l\in\mathbb{N}$ so that $P^l(M,T)$ continuously embeds into $C^k(M\times[0,T])$, we can conclude that $u$ actually belongs to $C^\infty(M\times[0,T])$.

Fixing $l\in\mathbb{N}$, we consider now the map

$$\mathcal{F}(u)=(u_0,u_t-L(u))=\Big(u(\,\cdot\,,0),u_t-Q^{ij}(u)\nabla^2_{ij}u-b(u)\Big)\,,$$

defined on $P^l(M,T)$, where $Q^{ij}(u)=Q^{ij}(p,t,u,\nabla u)$ and $b(u)=b(p,t,u,\nabla u)$. The image of the map $\mathcal{F}$ does not belong in general to $W^{2l-1,2}(M)\times P^{l-1}(M)$; this holds when $l\in\mathbb{N}$ is large enough and in this case $\mathcal{F}$ is actually $C^1$.

**Lemma A.3.4.** *Assume that $l>n/4+1$, then $u\in P^l(M,T)$ implies that $\nabla u$ belongs to $C^0(M\times[0,T])$. Moreover, $\mathcal{F}$ is a well-defined $C^1$ map from $P^l(M,T)$ to $W^{2l-1,2}(M)\times P^{l-1}(M,T)$.*

We postpone the proof of this lemma to the end of the section.

We fix $l\in\mathbb{N}$ such that the hypothesis of Lemma A.3.4 holds and we set $\widetilde{u}_0(p,t)=\sum_{m=0}^{l-1}a_m(p)\,t^m/m!$ for some functions $a_0,\dots,a_{l-1}\in C^\infty(M)$ to be determined later. Let $w\in P^l(M,T)$ be the unique solution of the linear problem

$$\begin{cases} w_t=Q^{ij}(p,t,\widetilde{u}_0,\nabla\widetilde{u}_0)\nabla^2_{ij}w+b(p,t,\widetilde{u}_0,\nabla\widetilde{u}_0),\\ w(\,\cdot\,,0)=u_0\,. \end{cases}$$

Such a solution exists by Proposition A.2.3 and it is smooth, as $u_0$ and $\widetilde{u}_0$ are smooth (see Remark A.3.3), hence we have

$$\mathcal{F}(w) = (w_0, w_t - L(w)) = \Big(u_0, (Q^{ij}(\widetilde{u}_0) - Q^{ij}(w))\nabla^2_{ij} w + b(\widetilde{u}_0) - b(w)\Big) = (u_0, f)\,,$$

where we set $f = (Q^{ij}(\widetilde{u}_0) - Q^{ij}(w))\nabla^2_{ij} w + b(\widetilde{u}_0) - b(w)$ which is a smooth function.

If we compute the differential $d\mathcal{F}_u$ of the map $\mathcal{F}$ at the "point" $w \in C^\infty(M \times [0,T]) \subset P^l(M,T)$, acting on $v \in P^l(M,T)$, we obtain

$$\begin{aligned} d\mathcal{F}_w(v) = \Big(v_0, v_t &- Q^{ij}(w)\nabla^2_{ij} v - \partial_w Q^{ij}(w) v \nabla^2_{ij} w \\ &- \partial_{w_k} Q^{ij}(w)\nabla_k v \nabla^2_{ij} w - \partial_w b(w) v - \partial_{w_k} b(w)\nabla_k v\Big)\,, \end{aligned} \tag{A.3.1}$$

where $v_0 = v(\,\cdot\,, 0)$.

Then, we can see that $d\mathcal{F}_w(v) = (z, h) \in W^{2l-1,2}(M) \times P^{l-1}(M,T)$ implies that $v$ is a solution of the linear system

$$\begin{cases} v_t - \widetilde{Q}^{ij}\nabla^2_{ij} v - \widetilde{R}^k \nabla_k v - \widetilde{S} v = h, \\ v(\,\cdot\,, 0) = z\,, \end{cases}$$

where $\widetilde{Q}^{ij} = Q^{ij}(w)$, $\widetilde{R}^k = \partial_{w_k} Q^{ij}(w)\nabla^2_{ij} w + \partial_{w_k} b(w)$ and $\widetilde{S} = \partial_w Q^{ij}(w)\nabla^2_{ij} w + \partial_w b(w)$ are smooth functions independent of $v$.

By Proposition A.2.3, for every $(z,h) \in W^{2l-1,2}(M) \times P^{l-1}(M,T)$ there exists a unique solution $v \in P^l(M,T)$ of this system, hence $d\mathcal{F}_w$ is a Hilbert space isomorphism and the inverse function theorem can be applied, as the map $\mathcal{F}$ is $C^1$ by Lemma A.3.4. Hence, the map $\mathcal{F}$ is a diffeomorphism of a neighborhood $U \subset P^l(M,T)$ of $w$ onto a neighborhood $V \subset W^{2l-1,2}(M) \times P^{l-1}(M,T)$ of $(u_0, f)$.

Getting back to the functions $a_m$, we claim that we can choose them such that $a_m = \partial_t^m w|_{t=0} \in C^\infty(M)$ for every $m = 0, \dots, l-1$.

We apply the following recurrence procedure. We set $a_0 = u_0 \in C^\infty(M)$ and, assuming we have defined $a_0, \dots, a_m$, we consider the derivative

$$\partial_t^{m+1} w\Big|_{t=0} = \partial_t^\ell [Q^{ij}(p, t, \widetilde{u}_0, \nabla\widetilde{u}_0)\nabla^2_{ij} w + b(p, t, \widetilde{u}_0, \nabla\widetilde{u}_0)]\Big|_{t=0}$$

and we see that the right-hand side contains time-derivatives at time $t = 0$ of $\widetilde{u}_0, \nabla\widetilde{u}_0$ and $\nabla^2_{ij} w$ only up to order $m$, hence it only depends on the functions $a_0, \dots, a_m$. Then, we define $a_{m+1}$ to be equal to such expression. Iterating up to $l-1$, the set of functions $a_0, \dots, a_{l-1}$ satisfies the claim.

Then, $a_m = \partial_t^m \widetilde{u}_0|_{t=0} = \partial_t^m w|_{t=0}$ and it easily follows by the "structure" of the function $f \in C^\infty(M \times [0,T])$, that we have $\partial_t^m f|_{t=0} = 0$ and $\nabla^j \partial_t^m f|_{t=0} = 0$ for any $0 \le \ell \le l-1$ and $j \in \mathbb{N}$.

We consider now for any $k \in \mathbb{N}$ the "translated" functions $f_k : M \times [0,T] \to \mathbb{R}$ given by

$$f_k(p,t) = \begin{cases} 0 & \text{if } t \in [0, 1/k], \\ f(p, t - 1/k) & \text{if } t \in (1/k, T]\,. \end{cases}$$

Since $f \in C^\infty(M \times [0,T])$ and $\nabla^j \partial_t^m f|_{t=0} = 0$ for every $0 \leq m \leq l-1$ and $j \in \mathbb{N}$, all the functions $\nabla^j \partial_t^m f_k \in C^0(M \times [0,T])$ for every $0 \leq m \leq l-1$ and $j \geq 0$, it follows easily that

$$\nabla^j \partial_t^m f_k \to \nabla^j \partial_t^m f \quad \text{ in } L^2(M \times [0,T]) \text{ for } 0 \leq m \leq l-1,\, j \geq 0\,,$$

hence $f_k \to f$ in $P^l(M,T)$.

Hence, there exists a function $\widetilde{f} \in P^{l-1}(M,T)$ such that $(u_0, \widetilde{f})$ belongs to the neighborhood $V$ of $\mathcal{F}(w)$, defined above and $\widetilde{f} = 0$ in $M \times [0,T']$ for some $T' \in (0,T]$. Since $\mathcal{F}|_U$ is a diffeomorphism between $U$ and $V$, we can find a function $u \in U$ such that $\mathcal{F}(u) = (u_0, \widetilde{f})$. Clearly such $u \in P^l(M,T)$ is a solution of problem (PDE) in $M \times [0,T']$. Since $u \in P^l(M,T')$ implies that $\nabla u \in C^0(M \times [0,T'])$, parabolic regularity implies that actually $u \in C^\infty(M \times [0,T'])$.

Now we get back to the original operator $L$, which we modified far from the initial time and initial datum in order to make it uniformly elliptic. The above argument shows that we can find a smooth solution $u$ in $M \times [0,T]$ for some $T > 0$ (we relabelled $T$ the time $T'$ found above). Such a solution is unique in every $P^l(M,T)$ for $l \in \mathbb{N}$ large enough (depending on the dimension of $M$). Indeed, by the Sobolev embeddings, $P^l(M,T)$ is a subspace of $C^k$, with $k$ growing with $l$, in particular for every $l \in \mathbb{N}$ large enough the solution we found is $C^2$ at least; this allows the use of the maximum principle in order to show that such solution is unique.

We finally prove the continuous dependence of a solution $u \in C^\infty(M \times [0,T])$ on its initial datum $u_0 = u(\,\cdot\,,0) \in C^\infty(M)$.

Fix any $l \in \mathbb{N}$ such that $l > n/4 + 1$, then by Lemma A.3.4 $u \in P^l(M,T)$ implies $\nabla u \in C^0(M \times [0,T])$. By the above argument, $u = (\mathcal{F}|_U)^{-1}(u_0, 0) \in P^l(M,T)$ where $\mathcal{F}|_U$ is a diffeomorphism of an open set $U \subset P^l(M,T)$ onto $V \subset W^{2l-1,2}(M) \times P^{l-1}(M,T)$, with $(u_0, 0) \in V$. Then, assuming that $u_{k,0} \to u_0$ in $C^\infty(M)$ as $k \to \infty$, we also have $u_{k,0} \to u_0$ in $W^{2l-1,2}(M)$, hence for $k$ large enough $(u_{k,0}, 0) \in V$ and there exists $u_k \in U$ such that $\mathcal{F}(u_k) = (u_{k,0}, 0)$. This is the unique solution in $P^l(M,T)$ (hence in $C^\infty(M \times [0,T])$ by parabolic bootstrap) with initial datum $u_{k,0}$. Moreover, since $\mathcal{F}|_U$ is a diffeomorphism, we have $u_k \to u$ in $P^l(M,T)$.

By uniqueness, we can repeat the same procedure for any $l \in \mathbb{N}$ satisfying the condition in Lemma A.3.4, concluding that $u_k \to u$ in $P^l(M,T)$ for every such $l \in \mathbb{N}$, hence in $C^\infty(M \times [0,T])$.

*Remark* A.3.5. Uniqueness can also be obtained by means of energy estimates based on Gårding's inequality for the operator $Q^{ij}(\,\cdot\,)\nabla^2_{ij}$, computing the ODE for the quantity $\int_M (w^2 + |\nabla w|^2)\, d\mu$ where $w = u - v$ and $u, v \in C^\infty(M \times [0,T])$ are a pair of solutions of problem (PDE).

We now prove Lemma A.3.4.

We need the following proposition which follows from standard arguments of parabolic interpolation theory, see [87, Theorem 2.3] and [90] for details.

**Proposition A.3.6.** *Let $u \in P^l(M,T)$ where $M$ is compact and $n$-dimensional. Then for $T > 0$ and $p, q \in \mathbb{N}$ with $p + 2q \leq 2l$, we have*

$$\|\partial_t^q \nabla^p u\|_{L^r(M\times[0,T])} \leq C\|u\|_{P^l(M,T)}\,, \ \textit{if} \quad \frac{1}{2} - \frac{2l-p-2q}{n+2} = \frac{1}{r} > 0\,, \tag{A.3.2}$$

$$\|\partial_t^q \nabla^p u\|_{L^r(M\times[0,T])} \leq C\|u\|_{P^l(M,T)}\,, \ \textit{if} \quad \frac{1}{2} - \frac{2l-p-2q}{n+2} = 0\,, \tag{A.3.3}$$

*for every $r \geq 1$.*
*Finally, $\partial_t^q \nabla^p u$ is continuous and*

$$\|\partial_t^q \nabla^p u\|_{C^0(M\times[0,T])} \leq C\|u\|_{P^l(M,T)}\,, \ \textit{if} \quad \frac{1}{2} - \frac{2l-p-2q}{n+2} < 0\,, \tag{A.3.4}$$

*where $C$ is a constant independent of $u \in P^l(M,T)$.*

*Proof of Lemma A.3.4.* The first claim follows immediately by the above proposition as the condition $l > n/4 + 1$ implies, choosing $p = 1$ and $q = 0$,

$$\frac{1}{2} - \frac{2l-p-2q}{n+2} = \frac{n+4-4l}{2(n+2)} < 0\,,$$

hence, $\nabla u$ is continuous.

To deal with the second claim, by simplicity we shall write $P^l = P^l(M,T)$, $L^q = L^q(M \times [0,T])$, $C^0 = C^0(M \times [0,T])$ etc., so that for instance $C^0(P^l; C^1)$ will denote the space of continuous maps from $P^l(M,T)$ to $C^1(M \times [0,T])$.
First we show that actually $\mathcal{F}(u) \in W^{2l-1,2} \times P^{l-1}$ when $u \in P^l$, hence the map $\mathcal{F}$ is well defined. The regularity of the first component follows from Lemma A.2.2 and linearity, the same holds for the term $u_t$ in the second component. Then we only need to prove that $\partial_t^m \nabla^k (Q^{ij}(u)\nabla^2_{ij} u + b(u))$ belongs to $L^2$ when $k + 2m = 2l - 2$ (actually, by looking at Definition A.2.1 of $P^{l-1}$, we should also check it when $2m + k < 2l - 2$ but this latter task is obviously easier).
Clearly, the "most difficult" term is $\partial_t^m \nabla^k (Q^{ij}(u)\nabla^2_{ij} u)$, moreover, when at least one time or space derivative apply to the $t$, $p$ and $u$ variables in $Q^{ij}(u)$, the resulting term will have a higher integrability than when all the derivatives go on the gradient term $\nabla u$ in $Q^{ij}(u)$.
Hence, for simplicity, we assume that $b(u) = 0$ and $Q^{ij}(u)\nabla^2_{ij} u = A^{ij}(\nabla u)\nabla^2_{ij} u$ that we will denote by $A(\nabla u)\nabla^2 u$, for some smooth tensor $A$. It will be clear by the following estimates that the other possible terms, when $b(u)$ is not zero and $Q^{ij}(u)$ depends also on the other variables, can be bounded analogously (actually more easily).
We underline that if all the derivatives go on the factor $\nabla^2 u$ of $A(\nabla u)\nabla^2 u$, the resulting term $A(\nabla u)\partial_t^m \nabla^k \nabla^2 u$ clearly belongs to $L^2$ as $A(\nabla u)$ is uniformly bounded (the gradient $\nabla u$ is continuous and $A$ is smooth), $k + 2m = 2l - 2$ and $u \in P^l$.

It can be easily proved by the Leibniz formula and induction that for every pair of integers $m$ and $k$ with $k+2m=2l-2$, the derivative $\partial_t^m\nabla^k(A(\nabla u)\nabla^2 u)$ is a finite sum of terms (the total number is bounded by a function of $l$) each one of the form

$$B(\nabla u)\prod_{p=1}^{2l}\prod_{q=0}^{l-1}\prod_{|\alpha|=p}(\partial_t^q\nabla_\alpha^p u)^{\sigma_{pq\alpha}} \tag{A.3.5}$$

where $\alpha$ is a multiindex of order $|\alpha|$, the exponents $\sigma_{pq\alpha}$ are nonnegative integers and $B$ stands for some smooth and bounded tensor. Hence, since $\nabla u$ is continuous, we estimate any of such terms as

$$|B(\nabla u)|\prod_{p=1}^{2l}\prod_{q=0}^{l-1}\prod_{|\alpha|=p}|\partial_t^q\nabla_\alpha^p u|^{\sigma_{pq\alpha}}\leq C\prod_{p=1}^{2l}\prod_{q=0}^{l-1}|\partial_t^q\nabla^p u|^{b_{pq}}$$

with $b_{pq}=\sum_{|\alpha|=p}\sigma_{pq\alpha}\in\mathbb{N}$ and nonnegative.
Moreover, it can be seen by induction on $l\in\mathbb{N}$ that the following formula holds:

$$\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1}} b_{pq}(p+2q-1)=2l-1\,. \tag{A.3.6}$$

Then,

$$\begin{aligned}\|\partial_t^m\nabla^k(A(\nabla u)\nabla^2 u)\|_{L^2}&\leq\sum C\Big\|\prod_{p=1}^{2l}\prod_{q=0}^{l-1}|\partial_t^q\nabla^p u|^{b_{pq}}\Big\|_{L^2}\\&\leq\sum C\Big(\int_{M\times[0,T]}\prod_{p=1}^{2l}\prod_{q=0}^{l-1}|\partial_t^q\nabla^p u|^{2b_{pq}}\,d\mu\,dt\Big)^{1/2},\end{aligned}$$

where the symbol of sum means that we are adding all the terms described by formula (A.3.5) above.
We now apply Proposition A.3.6 noticing that $(p+2q-1)$ is always positive; otherwise we must have $p=1$ and $q=0$ but the simple gradient $\nabla u$ cannot appear as a factor in the product formula (A.3.5), by the structure of $\partial_t^m\nabla^k(A(\nabla u)\nabla^2 u)$.
If for at least one pair $(p,q)$ with $b_{pq}\neq 0$ the derivative $\partial_t^q\nabla^p u$ is continuous by the embedding (A.3.4), we simply bound the relative factor with a constant and we modify the relative integer exponent $b_{pq}$ to be zero. It follows that we have also to modify the formula (A.3.6) to the inequality $\sum_{p=1}^{2l}\sum_{q=0}^{l-1}b_{pq}(p+2q-1)<2l-1$.
If at least one pair $(p,q)$ with $b_{pq}\neq 0$ satisfies $2(2l-p-2q)=n+2$, that is, we are in a critical case (A.3.3) of the embeddings, formula (A.3.6) gives

$$\sum_{\substack{p=1,\dots,2l\\ q=0,\dots,l-1\\ 2(2l-p-2q)\neq n+2}} b_{pq}(p+2q-1)<2l-1\,, \tag{A.3.7}$$

as we “dropped” at least one nonzero term $b_{pq}(p+2q-1)$.

Hence, either formula (A.3.7) holds or we did not "set to zero" any of the integers $b_{pq}$ and there are no factors in the critical cases of the embeddings. In this latter situation, either all the derivatives went on $\nabla^2 u$ and the resulting term $A(\nabla u)\partial_t^m \nabla^{k+2} u$ is bounded in $L^2$, or clearly there are at least two integers $b_{pq}$ which are nonzero.

We now estimate the previous integral with Hölder's inequality and the embeddings (A.3.2) for the factors with $2(2l-p-2q) \neq n+2$ and we choose a large $r_{pq}$ for any factor such that $2(2l-p-2q) = n+2$ (the critical cases), by the embeddings (A.3.3):

$$\begin{aligned}
\int_{M\times[0,T]} \prod_{p=1}^{2l}\prod_{q=0}^{l-1} |\partial_t^q \nabla^p u|^{2b_{pq}}\, d\mu\, dt &\le C \prod_{p=1}^{2l}\prod_{q=0}^{l-1} \left( \int_{M\times[0,T]} |\partial_t^q \nabla^p u|^{2b_{pq}/d_{pq}}\, d\mu\, dt \right)^{d_{pq}} \\
&= C \prod_{p=1}^{2l}\prod_{q=0}^{l-1} \left( \int_{M\times[0,T]} |\partial_t^q \nabla^p u|^{r_{pq}}\, d\mu\, dt \right)^{d_{pq}} \\
&\le C \|u\|_{P^l}^{\sum_{p=1}^{2l}\sum_{q=0}^{l-1} d_{pq} r_{pq}} \\
&= C \|u\|_{P^l}^{\sum_{p=1}^{2l}\sum_{q=0}^{l-1} 2b_{pq}}
\end{aligned}$$

where $\frac{1}{r_{pq}} = \frac{1}{2} - \frac{2l-p-2q}{n+2} > 0$ and $d_{pq} = 2b_{pq}/r_{pq}$.

This application of Hölder's inequality is justified if the sum of all the exponents $d_{pq}$ with $2(2l-p-2q) \neq n+2$ is less than 1, as we can choose the other $d_{pq}$ (associated to the critical cases) arbitrarily small.

In such a case we conclude that

$$\|\partial_t^m \nabla^k (A(\nabla u)\nabla^2 u)\|_{L^2} \le \sum C \|u\|_{P^l}^{\sum_{p=1}^{2l}\sum_{q=0}^{l-1} b_{pq}}$$

and we are done.

Hence, we now check such condition on the exponents $d_{pq}$ assuming that at least one of the integers $b_{pq}$ with $2(2l-p-2q) \neq n+2$ is nonzero, otherwise the conclusion is trivial.

$$\begin{aligned}
\sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} d_{pq} &= \sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} \frac{2b_{pq}}{r_{pq}} \\
&= \sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} 2b_{pq} \left( \frac{1}{2} - \frac{2l-p-2q}{n+2} \right) \\
&= \sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} 2b_{pq} \left( \frac{1}{2} - \frac{2l-1}{n+2} + \frac{p+2q-1}{n+2} \right)
\end{aligned}$$

$$= \sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)$$
$$+ \sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\frac{p+2q-1}{n+2}\,.$$

We now separate the two cases by the discussion above. If the strict inequality (A.3.7) holds we have

$$\sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} d_{pq} < \left[\sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1 \\ 2(2l-p-2q)\neq n+2}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)\right] + 2\frac{2l-1}{n+2} < 1\,,$$

as at least one of the integers $b_{pq}$ is not zero and since $\frac{1}{2}-\frac{2l-1}{n+2}<0$, by the hypothesis $l > n/4+1$.

If instead equality (A.3.6) holds, we have seen that at least two of the integers $b_{pq}$ are nonzero otherwise the conclusion is trivial, then for all the pairs $(p,q)$ with $b_{pq}>0$ there holds $2(2l-p-2q)\neq n+2$, hence,

$$\begin{aligned}\sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1}} d_{pq} &= \sum_{\substack{p=1,\dots,2l \\ q=0,\dots,l-1}} 2b_{pq}\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)+2\frac{2l-1}{n+2} \\ &\leq 4\left(\frac{1}{2}-\frac{2l-1}{n+2}\right)+2\frac{2l-1}{n+2} \\ &= 2-2\frac{2l-1}{n+2}\,,\end{aligned}$$

which is less than 1 again since $l > n/4+1$.

It remains to prove that $d\mathcal{F}\in C^0(P^l;L(P^l;P^{l-1}))$, where $L(P^l;P^{l-1})$ denotes the Banach space of bounded linear maps from $P^l$ into $P^{l-1}$. Again we assume by simplicity $b(u)=0$ and $Q(u)\nabla^2u=A(\nabla u)\nabla^2 u$, for some smooth tensor $A$ and we define $\mathcal{F}_A : P^l\to P^{l-1}$ given by $u\mapsto A(\nabla u)\nabla^2 u$.

We first claim that the Gateaux derivative

$$(u,v)\mapsto d\mathcal{F}_A(u)(v)=\frac{d}{dt}\mathcal{F}_A(u+\varepsilon v)\Big|_{\varepsilon=0}$$

belongs to $C^0(P^l\times P^l;P^{l-1})$. Indeed, $d\mathcal{F}_A(u)(v)$ is given by (see formula (A.3.1))

$$d\mathcal{F}_A(u)(v)=D(\nabla u)\nabla v\nabla^2 u + A(\nabla u)\nabla^2 v\,,$$

where $D$ is a smooth tensor and the procedure previously used to estimate $\mathcal{F}(u)$ can also be applied to any term $\partial_t^m\nabla^k(D(\nabla u)\nabla v\nabla^2 u)$ or $\partial_t^m\nabla^k(A(\nabla u)\nabla^2 v)$, since they can be expressed as a sum of terms similar to the ones of formula (A.3.5) with the only difference being that now in every term exactly one linear occurrence of $u$ is replaced by $v$.

It is then easy to conclude, since $v \in P^l$ like $u$, that we obtain the continuity of $(u,v) \mapsto d\mathcal{F}_A(u)(v)$ in the same way. This proves in particular that $d\mathcal{F}_A(u) \in L(P^l;P^{l-1})$.

In order now to show that $d\mathcal{F}_A \in C^0(P^l;L(P^l;P^{l-1}))$ we need to see that

$$\sup_{\|v\|_{P^l}\leq 1} \|d\mathcal{F}_A(\widetilde{u})(v) - d\mathcal{F}_A(u)(v)\|_{P^{l-1}} \to 0 \quad \text{as } \widetilde{u} \to u \text{ in } P^l\,.$$

Again, this estimate is similar to what we have already done. Indeed, by what we said above about the structure of the terms

$$\partial_t^m \nabla^k (D(\nabla u)\nabla v \nabla^2 u) \quad \text{and} \quad \partial_t^m \nabla^k (A(\nabla u)\nabla^2 v),$$

assuming for the sake of simplicity that there are no time derivatives, we have to show that, as $\widetilde{u} \to u$ in $P^l$,

$$\sup_{\|v\|_{P^l}\leq 1} \|B(\widetilde{u})\nabla^{i_1}\widetilde{u}\cdots\nabla^{i_j}\widetilde{u}\,\nabla^{i_{j+1}}v - B(u)\nabla^{i_1}u\cdots\nabla^{i_j}u\,\nabla^{i_{j+1}}v\|_{L^2} \to 0\,, \tag{A.3.8}$$

where $i_1 + \cdots + i_{j+1} = 2l + j$ (see formula (A.3.5) and condition (A.3.6)).

Adding and subtracting terms, one gets

$$\begin{aligned}
\Big| B(\widetilde{u})\nabla^{i_1}\widetilde{u}\cdots\nabla^{i_j}\widetilde{u}\,\nabla^{i_{j+1}}v &- B(u)\nabla^{i_1}u\cdots\nabla^{i_j}u\,\nabla^{i_{j+1}}v \Big| \\
&\leq \Big\{ |B(\widetilde{u}) - B(u)|\,|\nabla^{i_1}\widetilde{u}| \,\cdots\, |\nabla^{i_j}\widetilde{u}| \\
&\quad + |B(u)|\,|\nabla^{i_1}(\widetilde{u}-u)|\,|\nabla^{i_2}\widetilde{u}| \,\cdots\, |\nabla^{i_j}\widetilde{u}| \\
&\quad + \cdots + |B(u)|\,|\nabla^{i_1}u| \,\cdots\, |\nabla^{i_j}(\widetilde{u}-u)| \Big\}\, |\nabla^{i_{j+1}}v|\,.
\end{aligned}$$

Studying now the $L^2$ norm of this sum, the first term can be bounded as before and it goes to zero as $B(u)$ is continuous from $P^l$ to $L^\infty$. The $L^2$ norm of all the other terms, repeating step by step the previous estimates, using Hölder's inequality and embeddings (A.3.2)–(A.3.4), will be estimated by some product

$$C\|u\|^\alpha_{P^l}\|\widetilde{u}\|^\beta_{P^l}\|v\|^\gamma_{P^l}\|\widetilde{u}-u\|^\sigma_{P^l} \leq C\|u\|^\alpha_{P^l}\|\widetilde{u}\|^\beta_{P^l}\|\widetilde{u}-u\|^\sigma_{P^l}$$

for a constant $C$ and some nonnegative exponents $\alpha, \beta, \gamma, \sigma$ satisfying $\alpha+\beta+\gamma+\sigma \leq 1$ and $\sigma > 0$. Here we we used the fact that $\|v\|_{P^l} \leq 1$.

As $\widetilde{u} - u \to 0$ in $P^l$, this last product goes to zero in $L^2$, hence uniformly for $\|v\|_{P^l} \leq 1$ and inequality (A.3.8) follows, as claimed. The analysis of the estimates with mixed time/space derivatives is analogous.

Then, the Gateaux differential $d\mathcal{F}_A$ is continuous, which implies that it coincides with the Frechét differential, hence $\mathcal{F}_A \in C^1(P^l;P^{l-1})$. It follows that the map $\mathcal{F}$ is $C^1$ and we are done. $\square$

# Appendix B

# Interior Estimates of Ecker and Huisken

Let $M_t$ be a mean curvature flow of embedded hypersurfaces in $\mathbb{R}^{n+1}$ which are locally graphs of the functions $u(\,\cdot\,, t)$ over the hyperplane $e_{n+1}^{\perp} \approx \mathbb{R}^n$, where $u : \Omega \times [0, T] \to \mathbb{R}$ is a smooth function and $\Omega \subset \mathbb{R}^n$.
Let $\nu$ be the choice of the unit normal to the moving graphs such that the function $v = \langle \nu \,|\, e_{n+1} \rangle^{-1}$ is positive at time $t = 0$.

The estimates in the following series of theorems have been obtained by Ecker and Huisken in [38], see also [35] and [68].

**Theorem B.1.1.** *Let $R > 0$ and $x_0 \in \mathbb{R}^{n+1}$ be arbitrary and define*

$$\varphi(x, t) = R^2 - |x - x_0|^2 - 2nt\,.$$

*If $\varphi_+$ denotes the positive part of $\varphi$, we have the estimate*

$$v(x, t)\varphi_+(x, t) \leq \sup_{M_0} v\varphi_+$$

*as long as $v(x, t)$ is defined everywhere on the support of $\varphi_+$.*

**Theorem B.1.2.** *The gradient of the function $u$ satisfies the estimate*

$$\sqrt{1 + |\nabla u(y_0, t)|^2} \leq C_1(n) \sup_{B_R(y_0)} \sqrt{1 + |\nabla u_0|^2}$$
$$\times \exp\left[C_2(n)R^{-2} \sup_{s\in[0,T]} \Big( \sup_{r\in[0,T], y\in B_R(y_0)} u(y, r) - u(y_0, s)\Big)^2\right]$$

*where $t \in [0, T]$, $B_R(y_0)$ is a ball in $e_{n+1}^{\perp}$ and $u_0 = u(\,\cdot\,, 0)$.*

**Theorem B.1.3.** *Let $R > 0$ and $\theta \in [0, 1)$, then for $x_0 \in \mathbb{R}^{n+1}$ we have the estimate*

$$\sup_{K(x_0,t,\theta R^2)} |\mathrm{A}|^2 \leq C(n)(1 - \theta)^{-2} t^{-1} \sup_{0\leq s\leq t} \sup_{K(x_0,s,R^2)} v^4$$

*for all $t \in [0, T]$, where $K(x_0, t, \theta R^2) = \{x \in M_t \,|\, |x - x_0|^2 + 2nt \leq \theta R^2\}$.*

Working by induction, one gets the following general result.

**Theorem B.1.4.**

1. *For arbitrary* $R > 0$, $\theta \in [0, 1)$ *and* $m \in \mathbb{N}$, *we have the estimates*

$$\sup_{K(x_0,t,\theta R^2)} |\nabla^m \mathrm{A}|^2 \leq C_m(n) t^{-(m+1)}\,,$$

*where* $C_m(n)$ *depends on* $n, m, \theta$ *and* $\sup_{0\leq s\leq t} \sup_{K(x_0,s,R^2)} v^2$.

2. *In case of additional smoothness, the constants above can be replaced by other constants* $D_{m,k}(n)$, *depending on* $n, m, k, \theta$ *and*

$$\sup_{0\leq s\leq t} \sup_{K(x_0,s,R^2)} \sum_{i=0}^{m} |\nabla^i \mathrm{A}|^2$$

*obtaining the improved estimates*

$$\sup_{K(x_0,t,\theta R^2)} |\nabla^{m+k} \mathrm{A}|^2 \leq D_{m,k}(n) t^{-k}\,.$$

3. *In particular, choosing* $m = 0$ *and* $k = 1$,

$$\sup_{K(x_0,t,\theta R^2)} |\nabla \mathrm{A}|^2 \leq E(n)/t\,,$$

*where* $E(n)$ *depends on* $n$, $\theta$ *and* $\sup_{0\leq s\leq t} \sup_{K(x_0,s,R^2)} |\mathrm{A}|^2$.

Analogous estimates were obtained by Angenent [16] for evolving curves in the plane, by Altschuler [3] and Altschuler and Grayson [5] for curves in space.

*Remark* B.1.5. New estimates in the same spirit were obtained by Colding and Minicozzi in [28] where, for instance, it is proved a bound on the gradient (similar to the one in Theorem B.1.2) which depends only on the function $u$ at time $t = 0$ and not on its gradient.

# Appendix C

# Hamilton's Maximum Principle for Tensors

Let $V$ a vector bundle over a compact manifold $M$. Let $h$ be a fixed metric on $V$, $g$ a Riemannian metric on $M$ and $L$ a connection on $V$ compatible with $h$. Both $g$ and $L = \{L_{i\alpha}^{\beta}\}$ may depend on time $t$. We can form the Laplacian of a section $f$ of $V$ as the trace of the second covariant derivative with respect to $g$, using the connection $L$ on $V$ and the Levi–Civita connection on $TM$.

Let $U$ an open subset of $V$ and $\Psi(f)$ a vector field on $V$ tangent to the fibers. We consider the nonlinear PDE

$$\partial_t f = \Delta f + \Psi(f) \tag{PDE}$$

and the ODE

$$\partial_t f = \Psi(f)\,. \tag{ODE}$$

**Theorem C.1.1 (Hamilton [56, Section 4]).** *Let $X$ be a closed subset of $U \subset V$ invariant under parallel transport by the connection $L$ and such that every fiber of $X$ is convex.*

*If every solution of the ODE starting in a fiber of $X$ remains in $X$, then also any solution of the PDE remains in $X$.*

**Theorem C.1.2 (Hamilton [56, Section 8]).** *Let $f$ be a smooth section of $V$ satisfying $\partial_t f = \Delta f + \Psi(f)$. Let $Z(f)$ be a convex function on the bundle, invariant under parallel transport whose level curves $Z(f) \leq \lambda$ are preserved by the ODE. Then, the inequality $Z(f) \leq \lambda$ is also preserved by the PDE for any constant $\lambda$.*

*Moreover, if at time $t = 0$ at some point we have $Z(f) < \lambda$, then $Z(f) < \lambda$ everywhere on $M$ at every time $t > 0$.*

**Theorem C.1.3 (Hamilton [56, Section 8]).** *Let* $\mathrm{B}$ *be a symmetric bilinear form on $V$. Suppose that* $\mathrm{B}$ *satisfies the parabolic equation $\partial_t \mathrm{B} = \Delta \mathrm{B} + \Psi(\mathrm{B})$ where the matrix $\Psi(\mathrm{B}) \geq 0$ for all $\mathrm{B} \geq 0$.*

*Then, if* $\mathrm{B} \geq 0$ *at time* $t = 0$ *it remains nonnegative definite for* $t \geq 0$*. Moreover, there exists an interval* $0 < t < \delta$ *on which the rank of* $\mathrm{B}$ *is constant and the null space of* $\mathrm{B}$ *is invariant under parallel transport and invariant in time; finally it also lies in the null space of* $\Psi(\mathrm{B})$*.*

A good reference for these results is the book [27].

# Appendix D

# Hamilton's Matrix Li–Yau–Harnack Inequality in $\mathbb{R}^n$

**Theorem D.1.1.** *Let $u : \mathbb{R}^n \times (0, T] \to \mathbb{R}$ be a smooth positive solution of the heat equation such that for every $t \in (0, T]$ the function $u(\cdot, t)$ is bounded by some constant $C(t) > 0$ (possibly unbounded as $t \to 0$).*
*Then, the Hamilton quadratic*

$$H_{ij} = \nabla^2_{ij} u - \frac{\nabla_i u \nabla_j u}{u} + \frac{u}{2t}\delta_{ij}$$

*is nonnegative definite for every $x \in \mathbb{R}^n$ and $t > 0$.*

*Proof.* It is well known that being a solution of the heat equation, the boundedness of $u$ in space implies that also $|\nabla u|$ and $|\nabla^2 u|$ are bounded in space by some constant depending only on $t$, that we still call $C(t)$.
We suppose for the moment that these constants $C(t)$ are uniformly bounded from above by $C < +\infty$ and that $u > \theta > 0$ for some $\theta \in \mathbb{R}$.
We set, for some positive constants $\varepsilon$ and $A$,

$$H^\varepsilon_{ij} = H_{ij} + \frac{\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}\delta_{ij} \,.$$

With this choice we see that

$$H^\varepsilon_{ij}(x,t) \geq -\left|\nabla^2_{ij} u - \frac{\nabla_i u \nabla_j u}{u}\right|\delta_{ij} + \frac{\theta}{2t}\delta_{ij} \geq -(C + C^2/\theta)\delta_{ij} + \frac{\theta}{2t}\delta_{ij}\,, \qquad \text{(D.1)}$$

and since the last term goes to $+\infty$ when $t \to 0$, for $t$ small $H^\varepsilon_{ij}$ is uniformly positive definite.

After a straightforward computation (see [58] or [93]) we get

$$\partial_t H_{ij} \geq \Delta H_{ij} + \frac{2}{u}H^2_{ij} - \frac{2}{t}H_{ij}\,,$$

hence, as $\frac{e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}$ solves the heat equation in $\mathbb{R}^n \times [0, A)$,

$$\begin{aligned}(\partial_t - \Delta)H^\varepsilon_{ij} &\geq \frac{2}{u}H^2_{ij} - \frac{2}{t}H_{ij}\\ &= \frac{2}{u}(H^\varepsilon_{ij})^2 - \frac{2}{t}H^\varepsilon_{ij} - \frac{4\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{u(A-t)^{n/2}}H^\varepsilon_{ij} + \frac{2\varepsilon^2 e^{\frac{|x|^2}{2(A-t)}}}{u(A-t)^n}\delta_{ij} + \frac{2\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{t(A-t)^{n/2}}\delta_{ij}\,.\end{aligned}$$

For every $t \in (0, A)$ the form $H^\varepsilon_{ij}$ gets its smallest eigenvalue at some point $x_0 \in \mathbb{R}^n$ as the term $H^\varepsilon_{ij}$ is bounded and the term $\frac{\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}\delta_{ij}$ goes to $+\infty$ as $|x| \to +\infty$. If at some $t_0 < A/2$ the smallest eigenvalue of $H^\varepsilon_{ij}$ at $x_0 \in \mathbb{R}^n$ is zero and $\{v^i\}$ is a relative unit eigenvector, at such point $x_0$ we have

$$(\partial_t - \Delta)(H^\varepsilon_{ij}v^iv^j) \geq \frac{2\varepsilon^2 e^{\frac{|x|^2}{2(A-t)}}}{u(A-t)^n} + \frac{2\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{t(A-t)^{n/2}} > 0\,.$$

Thus, considering the *first point* in space-time $(x_0, t_0)$, with $t_0 < A/2$ such that $H^\varepsilon_{ij}$ is not positive definite (such a first point must exist by the estimate (D.1)) and considering a relative unit eigenvector $\{v^i\}$, we have that the function $h(x,t) = H^\varepsilon_{ij}(x,t)v^iv^j$ is positive for every $(x,t)$ with $t < t_0$, hence $(\partial_t - \Delta)h(x_0,t_0) \leq 0$, which is in contradiction with the previous estimate.

Hence, for every $\varepsilon > 0$, the matrix $H^\varepsilon_{ij}(x,t)$ is positive definite for every $t \in (0, A/2)$ and $x \in \mathbb{R}^n$.

Since $H^\varepsilon_{ij}(x,t) = H_{ij}(x,t) + \frac{\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}\delta_{ij}$, sending $\varepsilon$ to zero and $A$ to $+\infty$, we conclude that for every $x \in \mathbb{R}^n$ and $t > 0$, the Harnack quadratic $H_{ij}$ is nonnegative definite.

Let now $u$ be general (not uniformly bounded from below by some $\theta > 0$), given $\varepsilon > 0$ we consider the positive solution of the heat equation $w(x,s) = \varepsilon + u(x, s+\varepsilon) : \mathbb{R}^n \times [0, T-\varepsilon) \to \mathbb{R}$. As $w$, $|\nabla w|$ and $|\nabla^2 w|$ are uniformly bounded respectively by $\sup_{\mathbb{R}^n\times[\varepsilon,T]} u$, $\sup_{\mathbb{R}^n\times[\varepsilon,T]} |\nabla u|$ and $\sup_{\mathbb{R}^n\times[\varepsilon,T]} |\nabla^2 u|$, by what we proved we conclude

$$0 \leq \nabla^2_{ij}w - \frac{\nabla_i w \nabla_j w}{w} + \frac{w}{2s}\delta_{ij} = \nabla^2_{ij}u - \frac{\nabla_i u \nabla_j u}{u+\varepsilon} + \frac{u+\varepsilon}{2(t-\varepsilon)}\delta_{ij}$$

for every $x \in \mathbb{R}^n$ and $t \in (\varepsilon, T]$, where we substituted $s + \varepsilon = t$.
As $u$ is positive at every point, sending $\varepsilon$ to zero we have the thesis. $\square$

For more details on this topic consult the book [27].

# Appendix E

# Abresch and Langer Classification of Homothetically Shrinking Closed Curves

**(In collaboration with Annibale Magni)**

We briefly discuss here the classification of Abresch and Langer in [1] of the homothetically shrinking closed curves in the plane, that is, satisfying the structural equation $k + \langle x \,|\, \nu \rangle = 0$.

We have seen in Proposition 3.4.1 that among the curves with $k + \langle x \,|\, \nu \rangle = 0$, the only embedded, complete and connected ones are the lines through the origin and the unit circle.

If we now do not assume the embeddedness, we have to deal either with a smooth, complete immersion of $\mathbb{S}^1$ or of $\mathbb{R}$ possibly with self-intersections. In the first case, the curve is closed and compact, in the second case we can see that the initial part of the analysis in the proof of Proposition 3.4.1 still holds, hence, either the curve is a line through the origin of $\mathbb{R}^2$ or the curvature is everywhere positive and bounded from above, which implies that the whole curve is bounded. Then, if we only assume an estimate of the length of the curve (which holds by Lemma 3.2.7 and boundedness if the curve is a blow-up limit), by the completeness it follows that the curve is closed.

Hence, we concentrate only on closed curves. As we said, $k > 0$, the equations (3.4.1) hold and the quantity $k_\theta^2 + k^2 - \log k^2$ is constant along the curve, equal to some constant $E$ which must be larger than one (otherwise we are dealing with the unit circle). Again, the curve is symmetric with respect to the critical points of the curvature, which are all nondegenerate, isolated and finite. Hence, the curvature $k(\theta)$ is oscillating between its maximum and its minimum with some period $T > 0$. If we exclude the unit circle, such period must be an integer fraction (at least of a factor 2 by the four-vertex theorem, see the proof of Proposition 3.4.1) of an integer multiple (at least 2, otherwise we are dealing with the unit circle) of $2\pi$, that is, $T = 2m\pi/n$ with $n, m \geq 2$.

*Notice that there are two parameters here, the rotation number of the closed curve and the number of critical points of the curvature.*

Suppose that $k_{\min} < k_{\max}$ are these two consecutive critical values of $k$; it follows that they are two distinct positive zeroes of the function $E + \log k^2 - k^2$ when $E > 1$ with $0 < k_{\min} < 1 < k_{\max}$.
We have that the change $\Delta\theta$ in the angle $\theta$ along the piece of curve delimited by two consecutive points where the curvature assumes the values $k_{\min}$ and $k_{\max}$, is equal to the semiperiod $T/2$. Then, the analysis reduces to understanding what are the admissible $T$.
Such quantity $\Delta\theta$ is given by the integral

$$I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}\,.$$

Abresch and Langer (and also Epstein and Weinstein) by studying the behavior of this integral were able to classify *all* the immersed closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$. These form a family of curves indexed by two parameters called *Abresch–Langer curves*, see [1] for a detailed description.
We now state and partially prove the main properties of the integral $I(E)$ needed in such analysis.
It should be noticed that, by the discussion about the period $T$, the last statement in the next proposition implies Proposition 3.4.1.

**Proposition E.1.1.** *The function $I : (1, +\infty) \to \mathbb{R}$ satisfies*

1. $\lim_{E\to 1^+} I(E) = \pi/\sqrt{2}$,
2. $\lim_{E\to +\infty} I(E) = \pi/2$,
3. *$I(E)$ is monotone nonincreasing.*

*As a consequence $I(E) > \pi/2$.*

*Proof.* Notice that the study of the quantity $I(E)$ is equivalent to the study of the semiperiod for the one-dimensional Hamiltonian system with Hamiltonian function given by $H(k_\theta, k) = (k_\theta^2 + k^2 - \log k^2)/2$.

(1) The global minimum $1/2$ of the strictly convex potential $V(k) = (k^2 - \log k^2)/2$ is assumed at $k = 1$ and the limiting value for the period of the Hamiltonian system when $E \to 1^+$ is equal to the period of the corresponding linearized system (see [44, Chapter 12]). The linearized Hamiltonian is $H_L(\hat{k}_\theta, \hat{k}) = \hat{k}_\theta^2/2 + \hat{k}^2 + 1/2$ for the new variable $\hat{k} = k - 1$, which gives the equation $\hat{k}_{\theta\theta} = -2\hat{k}$ for $\hat{k}$. As any solution of this last ODE is clearly $\sqrt{2}\pi$-periodic, its semiperiod is equal to $\pi/\sqrt{2}$.

(2) As $0 < k_{\min} < 1 < k_{\max}$ for $E > 1$, we can write

$$I(E) = \int_{k_{\min}}^{1} \frac{dk}{\sqrt{E - k^2 + \log k^2}} + \int_{1}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}} = I_-(E) + I_+(E)\,.$$

We want to prove that $\lim_{E\to+\infty} I_-(E) = 0$ and $\lim_{E\to+\infty} I_+(E) = \pi/2$. Introducing the variable $w = k/k_{\min}$ the first integral becomes

$$I_-(E) = k_{\min}\int_1^{1/k_{\min}} \frac{dw}{\sqrt{k_{\min}^2(1-w^2)+\log w^2}}\,.$$

Notice that, given a real number $0<\alpha<1$, it is always possible to find $\widetilde{k}(\alpha)$ such that $|k_{\min}(1-w^2)| \le \alpha|\log w^2|$ with $w\in[1,1/k_{\min}]$ and $k_{\min}\le\widetilde{k}$. Fixing then such an $\alpha$, we have

$$\begin{aligned}
0 \le I_-(E) \;&\le \frac{k_{\min}}{\sqrt{1-\alpha}}\int_1^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \\
&\le \frac{k_{\min}}{\sqrt{1-\alpha}}\Big(\int_1^{n} \frac{dw}{\sqrt{2\log w}} + \int_n^{1/\sqrt{k_{\min}}} \frac{dw}{\sqrt{2\log w}} + \int_{1/\sqrt{k_{\min}}}^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}}\Big) \\
&\le k_{\min}(C_1 + C_2/\sqrt{k_{\min}} + o_{k_{\min}}(1)/k_{\min})\,,
\end{aligned}$$

hence, the claim on $I_-(E)$ follows.
Regarding $I_+(E)$, we proceed in a similar way by changing again the integration variable to $w = k/k_{\max}$. In this way we obtain

$$\begin{aligned}
\lim_{E\to+\infty} I_+(E) &= \lim_{E\to+\infty}\int_{1/k_{\max}}^{1} \frac{dw}{\sqrt{1-w^2+\frac{2\log w}{k_{\max}^2}}} \\
&= \lim_{E\to+\infty}\int_0^1 \chi_{[1/k_{\max},1]}\frac{dw}{\sqrt{1-w^2+\frac{2\log w}{k_{\max}^2}}} \\
&= \pi/2\,,
\end{aligned}$$

where in the last equality we applied the dominated convergence theorem.

(3) See the original paper of Abresch and Langer [1] or the general result by Zevin and Pinsky in [125]. $\square$

# Appendix F

# Important Results without Proof in the Book

- Interior estimates of Ecker and Huisken in [38] (Appendix B).
- Hamilton's maximum principle for tensors in [56, Sections 4 and 8] (Appendix C).
- Convexity estimates of Huisken and Sinestrari in [73] (Theorem 4.2.1).
- Regularity theorem of White in [123] (Theorem 3.2.22).
- Monotonicity of $I(E)$ by Abresch and Langer in [1] (Appendix E).
- Pinching theorem of Hamilton in [60] (Theorem 4.1.8).
- Sturmian theorem of Angenent in [16, Proposition 1.2] and [13] (Proposition 4.3.1).
- Hamilton's Li–Yau–Harnack estimate for mean curvature flow in [62] (Theorems 4.4.1 and 4.4.2).

# Bibliography


[1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom. **23** (1986), no. 2, 175–196.

[2] F.J. Almgren, J.E. Taylor, and L. Wang, *Curvature driven flows: a variational approach*, SIAM J. Control Opt. **31** (1993), 387–438.

[3] S.J. Altschuler, *Singularities of the curve shrinking flow for space curves*, J. Diff. Geom. **34** (1991), no. 2, 491–514.

[4] S.J. Altschuler, S. Angenent, and Y. Giga, *Mean curvature flow through singularities for surfaces of rotation*, J. Geom. Anal. **5** (1995), no. 3, 293–358.

[5] S.J. Altschuler and M. Grayson, *Shortening space curves and flow through singularities*, J. Diff. Geom. **35** (1992), 283–298.

[6] L. Ambrosio, *Geometric evolution problems, distance function and viscosity solutions*, Calculus of variations and partial differential equations (Pisa, 1996), Springer-Verlag, Berlin, 2000, pp. 5–93.

[7] L. Ambrosio and H.M. Soner, *A level set approach to the evolution of surfaces of any codimension*, J. Diff. Geom. **43** (1996), 693–737.

[8] B. Andrews, *Flow of hypersurfaces by curvature functions*, Workshop on theoretical and numerical aspects of geometric variational problems (Canberra, 1990), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 26, Austral. Nat. Univ., Canberra, 1991, pp. 1–10.

[9] ———, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171.

[10] ———, *Non-collapsing in mean-convex mean curvature flow*, unpublished work, 2009.

[11] B. Andrews and C. Baker, *Mean curvature flow of pinched submanifolds to spheres*, J. Diff. Geom. **85** (2010), no. 3, 357–395.

[12] B. Andrews and P. Bryan, *Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem*, ArXiv Preprint Server – http://arxiv.org, 2009.

[13] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79–96.

[14] ———, *Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature*, Ann. of Math. (2) **132** (1990), no. 3, 451–483.

[15] ———, *On the formation of singularities in the curve shortening flow*, J. Diff. Geom. **33** (1991), 601–633.

[16] ———, *Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions*, Ann. of Math. (2) **133** (1991), no. 1, 171–215.

[17] ———, *Shrinking doughnuts*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), Progr. Nonlinear Differential Equations Appl., vol. 7, Birkhäuser Boston, Boston, MA, 1992, pp. 21–38.

[18] S. Angenent, T. Ilmanen, and D.L. Chopp, *A computed example of non-uniqueness of mean curvature flow in* $\mathbb{R}^3$, Comm. Partial Differential Equations **20** (1995), no. 11-12, 1937–1958.

[19] S. Angenent and J.J.L. Velázquez, *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math. **482** (1997), 15–66.

[20] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, 1998.

[21] K.A. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, NJ, 1978.

[22] B.-L. Chen and L. Yin, *Uniqueness and pseudolocality theorems of the mean curvature flow*, Comm. Anal. Geom. **15** (2007), no. 3, 435–490.

[23] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom. **33** (1991), 749–786.

[24] D.L. Chopp, *Computation of self-similar solutions for mean curvature flow*, Experiment. Math. **3** (1994), no. 1, 1–15.

[25] K.-S. Chou and X.-P. Zhu, *Shortening complete plane curves*, J. Diff. Geom. **50** (1998), no. 3, 471–504.

[26] ———, *The curve shortening problem*, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[27] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications. Part II. Analytic aspects*, Mathematical Surveys and Monographs, vol. 144, American Mathematical Society, Providence, RI, 2008.

[28] T.H. Colding and W.P. Minicozzi II, *Sharp estimates for mean curvature flow of graphs*, J. Reine Angew. Math. **574** (2004), 187–195.

[29] ———, *Generic mean curvature flow I; generic singularities*, ArXiv Preprint Server – http://arxiv.org, 2009.

[30] T.H. Colding and B. Kleiner, *Singularity structure in mean curvature flow of mean-convex sets*, Electron. Res. Announc. Amer. Math. Soc. **9** (2003), 121–124 (electronic).

[31] P. Daskalopoulos, R.S. Hamilton, and N. Sesum, *Classification of compact ancient solutions to the curve shortening flow*, J. Diff. Geom. **84** (2010), no. 3, 455–464.

[32] C. Dellacherie and P.-A. Meyer, *Probabilities and potential*, North-Holland Mathematics Studies, vol. 29, North-Holland Publishing Co., Amsterdam, 1978.

[33] D.M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Diff. Geom. **18** (1983), no. 1, 157–162.

[34] K. Ecker, *Lectures on geometric evolution equations*, Instructional Workshop on Analysis and Geometry, Part II (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., Canberra, 1996, pp. 79–107.

[35] ______, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004.

[36] ______, *A formula relating entropy monotonicity to Harnack inequalities*, Comm. Anal. Geom. **15** (2007), no. 5, 1025–1061.

[37] K. Ecker and G. Huisken, *Mean curvature flow of entire graphs*, Ann. of Math. (2) **130** (1989), 453–471.

[38] ______, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.

[39] M. Eminenti, *Alcune proprietà della funzione distanza da una sottovarietà e una congettura di Ennio De Giorgi*, unpublished work, 2004.

[40] C.L. Epstein and M.I. Weinstein, *A stable manifold theorem for the curve shortening equation*, Comm. Pure Appl. Math. **40** (1987), no. 1, 119–139.

[41] L.C. Evans and R.F. Gariepy, *Lectures notes on measure theory and fine properties of functions*, CRC Press, Ann Arbor, 1992.

[42] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Diff. Geom. **33** (1991), 635–681.

[43] ______, *Motion of level sets by mean curvature II*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 321–332.

[44] A. Fasano and S. Marmi, *Analytical dynamics: an introduction*, Oxford Graduate Texts, 2006.

[45] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1964.

[46] M. Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Math. J. **50** (1983), no. 4, 1225–1229.

[47] ______, *Curve shortening makes convex curves circular*, Invent. Math. **76** (1984), 357–364.

[48] M. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–95.

[49] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, 1990.

[50] Y. Giga, *Surface evolution equations. A level set approach*, Monographs in Mathematics, vol. 99, Birkhäuser, Basel, 2006.

[51] M.A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Diff. Geom. **26** (1987), 285–314.

[52] ______, *A short note on the evolution of surfaces via mean curvature*, Duke Math. J. **58** (1989), 555–558.

[53] ______, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989), 71–111.

[54] C. Gui, H. Jian, and H. Ju, *Properties of translating solutions to mean curvature flow*, Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 441–453.

[55] R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), no. 2, 255–306.

[56] ______, *Four-manifolds with positive curvature operator*, J. Diff. Geom. **24** (1986), no. 2, 153–179.

[57] ______, *The Harnack estimate for the Ricci flow*, J. Diff. Geom. **37** (1993), no. 1, 225–243.

[58] ______, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), no. 1, 113–126.

[59] ______, *Monotonicity formulas for parabolic flows on manifolds*, Comm. Anal. Geom. **1** (1993), no. 1, 127–137.

[60] ______, *Convex hypersurfaces with pinched second fundamental form*, Comm. Anal. Geom. **2** (1994), no. 1, 167–172.

[61] ______, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 7–136.

[62] ______, *The Harnack estimate for the mean curvature flow*, J. Diff. Geom. **41** (1995), no. 1, 215–226.

[63] ______, *Isoperimetric estimates for the curve shrinking flow in the plane*, Modern methods in complex analysis (Princeton, NJ, 1992), Princeton University Press, NJ, 1995, pp. 201–222.

[64] ______, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), no. 1, 1–92.

[65] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984), 237–266.

[66] ______, *Nonparametric mean curvature evolution with boundary conditions*, J. Diff. Eqs. **77** (1989), no. 2, 369–378.

[67] ______, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. **31** (1990), 285–299.

[68] ______, *Local and global behaviour of hypersurfaces moving by mean curvature*, Proc. Sympos. Pure Math **54** (1993), 175–191.

[69] ______, *Lectures on geometric evolution equations*, Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991), Int. Press, Cambridge, MA, 1997, pp. 117–143.

[70] ______, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), 127–133.

[71] ______, *Evolution of hypersurfaces by their curvature in Riemannian manifolds*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 349–360 (electronic).

[72] G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Springer-Verlag, Berlin, 1999, pp. 45–84.

[73] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), no. 1, 45–70.

[74] ______, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations **8** (1999), no. 1, 1–14.

[75] ______, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. **175** (2009), no. 1, 137–221.

[76] N. Hungerbühler and K. Smoczyk, *Soliton solutions for the mean curvature flow*, Differential Integral Equations **13** (2000), no. 10–12, 1321–1345.

[77] T. Ilmanen, *Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature*, J. Diff. Geom. **38** (1993), 417–461.

[78] ______, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc., vol. 108(520), AMS, 1994.

[79] ______, *Singularities of mean curvature flow of surfaces*, http://www.math.ethz.ch/~ilmanen/papers/sing.ps, 1995.

[80] ______, *Lectures on mean curvature flow and related equations*, http://www.math.ethz.ch/~ilmanen/papers/notes.ps, 1998.

[81] N. Ishimura, *Shape of spirals*, Tohoku Math. J. (2) **50** (1998), no. 2, 197–202.

[82] E.M. Landis, *Second order equations of elliptic and parabolic type*, translations of Mathematical Monographs, vol. 171, American Mathematical Society, Providence, RI, 1998, translated from the 1971 Russian original by Tamara Rozhkovskaya, with a preface by Nina Ural'tseva.

[83] J. Langer, *A compactness theorem for surfaces with $L_p$-bounded second fundamental form*, Math. Ann. **270** (1985), 223–234.

[84] N.Q. Le and N. Sesum, *On the extension of the mean curvature flow*, ArXiv Preprint Server – http://arxiv.org, 2009.

[85] ———, *The mean curvature at the first singular time of the mean curvature flow*, Ann. Inst. H. Poicaré Anal. Non Linéaire **27** (2010), no. 6, 1441–1459.

[86] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201.

[87] J.L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972.

[88] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser, Basel, 1995.

[89] L. Lusternik and L. Schnirelman, *Sur le problème de trois géodésiques fermées sur les surfaces de genre* 0, C. R. Acad. Sci. Paris **189** (1929), 269–271.

[90] C. Mantegazza and L. Martinazzi, *A note on quasilinear parabolic equations on manifolds*, ArXiv Preprint Server – http://arxiv.org, to appear on Ann. Sc. Norm. Sup. Pisa, 2010.

[91] M. Marcus and L. Lopes, *Inequalities for symmetric functions and Hermitian matrices*, Canad. J. Math. **9** (1957), 305–312.

[92] S. Mukhopadhyaya, *New methods in the geometry of a plane arc*, Bull. Calcutta Math. Soc. **1** (1909), 21–27.

[93] R. Müller, *Differential Harnack inequalities and the Ricci flow*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2006.

[94] W.M. Mullins, *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. **27** (1956), 900–904.

[95] X.H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. I*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 1683–1701.

[96] ———, *Translating tridents*, Comm. Partial Differential Equations **34** (2009), no. 1–3, 257–280.

[97] ———, *Complete embedded self-translating surfaces under mean curvature flow*, ArXiv Preprint Server – http://arxiv.org, 2010.

[98] ———, *Construction of complete embedded self-similar surfaces under mean curvature flow. II*, Adv. Differential Equations **15** (2010), no. 5–6, 503–530.

[99] S. Osher and J. Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations*, J. Comput. Phys. **79** (1988), no. 1, 12–49.

[100] R. Osserman, *The four-or-more vertex theorem*, Amer. Math. Monthly **92** (1985), no. 5, 332–337.

[101] M. Paolini and C. Verdi, *Asymptotic and numerical analyses of the mean curvature flow with a space-dependent relaxation parameter*, Asymptotic Anal. **5** (1992), no. 6, 553–574.

[102] A. Polden, *Curves and Surfaces of Least Total Curvature and Fourth-Order Flows*, Ph.D. thesis, Mathematisches Institut, Univ. Tübingen, 1996, Arbeitsbereich Analysis Preprint Server – Univ. Tübingen, http://poincare.mathematik.uni-tuebingen.de/mozilla/home.e.html.

[103] W. Sheng and X.-J. Wang, *Singularity profile in the mean curvature flow*, Methods Appl. Anal. **16** (2009), no. 2, 139–155.

[104] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Diff. Geom. **30** (1989), no. 1, 223–301.

[105] L. Simon, *Lectures on geometric measure theory*, Proc. Center Math. Anal., vol. 3, Australian National University, Canberra, 1983.

[106] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.

[107] C. Sinestrari, *Singularities of mean curvature flow and flow with surgeries*, Surveys in differential geometry. Vol. XII. Geometric flows, vol. 12, Int. Press, Somerville, MA, 2008, pp. 303–332.

[108] K. Smoczyk, *Starshaped hypersurfaces and the mean curvature flow*, Manuscripta Math. **95** (1998), no. 2, 225–236.

[109] H.M. Soner, *Motion of a set by the curvature of its boundary*, J. Diff. Eqs. **101** (1993), no. 2, 313–372.

[110] H.M. Soner and P.E. Souganidis, *Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature*, Comm. Partial Differential Equations **18** (1993), 859–894.

[111] P.E. Souganidis, *Front propagation: theory and applications*, Viscosity solutions and applications (Montecatini Terme, 1995), Lect. Notes in Math., vol. 1660, Springer-Verlag, Berlin, 1997, pp. 186–242.

[112] A. Stahl, *Convergence of solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differential Equations **4** (1996), no. 5, 421–441.

[113] ———, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differential Equations **4** (1996), no. 4, 385–407.

[114] N. Stavrou, *Selfsimilar solutions to the mean curvature flow*, J. Reine Angew. Math. **499** (1998), 189–198.

[115] A. Stone, *A density function and the structure of singularities of the mean curvature flow*, Calc. Var. Partial Differential Equations **2** (1994), 443–480.

[116] ———, *Singular and Boundary Behaviour in the Mean Curvature Flow of Hypersurfaces*, Ph.D. thesis, Stanford University, 1994.

[117] M.-T. Wang, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, Invent. Math. **148** (2002), no. 3, 525–543.

[118] ———, *The mean curvature flow smoothes Lipschitz submanifolds*, Comm. Anal. Geom. **12** (2004), no. 3, 581–599.

[119] X.-J. Wang, *Convex solutions to the mean curvature flow*, ArXiv Preprint Server – http://arxiv.org, to appear on Ann. of Math., 2004.

[120] B. White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **13** (2000), no. 3, 665–695 (electronic).

[121] ———, *Evolution of curves and surfaces by mean curvature*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 2002, pp. 525–538.

[122] ———, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), no. 1, 123–138 (electronic).

[123] ———, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.

[124] H. Wu, *Manifolds of partially positive curvature*, Indiana Univ. Math. J. **36** (1987), no. 3, 525–548.

[125] A.A. Zevin and M.A. Pinsky, *Monotonicity criteria for an energy-period function in planar Hamiltonian systems*, Nonlinearity **14** (2001), no. 6, 1425–1432.

[126] X.-P. Zhu, *Lectures on mean curvature flows*, AMS/IP Studies in Advanced Mathematics, vol. 32, American Mathematical Society, Providence, RI, 2002.

# Index